Plasticity and Deformation Process

Response of materials to stress

Elastic deformation

The aim of Plasticity and Deformation Process course is to develop a clear understanding of the macroscopic behavior of materials subjected to a deformation process, the underlying microstructural mechanisms and the effects of various physical and chemical process parameters on the response of the material.

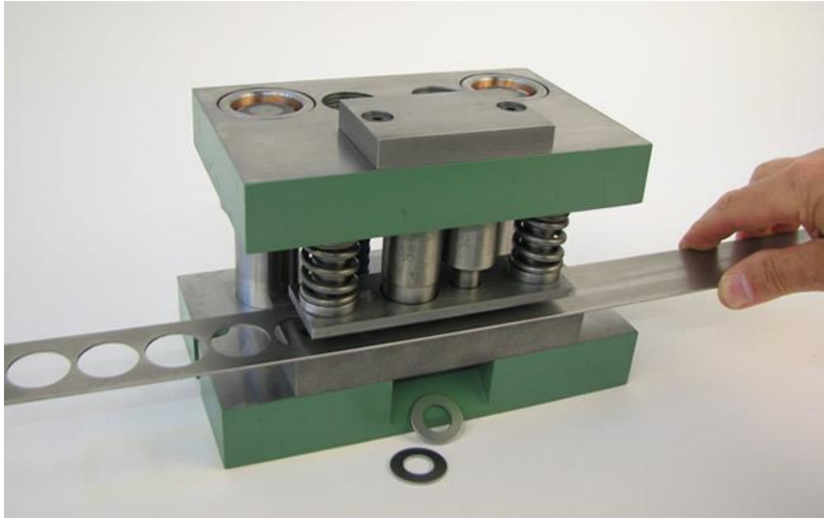
We need to examine materials from two points of view for this aim:

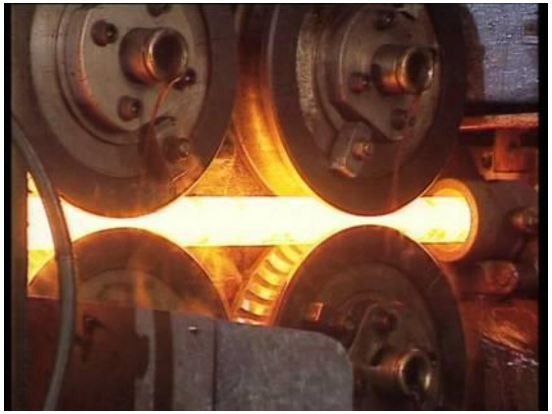
1. Chemical physics, where the focus is on the molecular interactions that result in the observed mechanical behavior
2. Engineering view, where the molecular nature of matter is ignored and the laws of elasticity, plasticity, viscoelasticity and viscoplasticity are used to explain the observed mechanical behavior

By these approaches we can find solutions to the following questions:

- ✓ How much will a material deform under a given load?
- ✓ Is the deformation permanent or will the material return to its original shape after the load is removed?
- ✓ How big a load can the material carry before it breaks?
- ✓ How does the material behave under impact, as opposed to static loads?

We will apply the plasticity theory to industrial deformation processes for **metals** and **polymers**





Remembering the basic definitions

Load – force applied uniformly on an object, unit: Newton or kg-force

Deformation – the response of the object to the applied load by changing its shape and size, unit: m

Stress – force applied on unit area of an object, or pressure, unit: Newton/m^2 or pascal

Strain – the ratio of the change in size of the material to the initial size, no unit

Mechanical energy – stress applied through a volume of an object, unit: Newton*m or Joule

Would you expect a steel rod with a diameter of 1 cm to carry more load than a steel rod with a diameter of 1 mm?

Would you expect a steel rod with a length of 1 m to deform more under a constant load than a steel rod with a length of 10 cm?

Now consider a rubber fiber with the same dimensions as the steel rod (diameter= 1 cm, L= 1m)
Would it carry more load?
Would it deform more under the same load?

In order to compare the mechanical properties of different materials, we must take out the effect of shape and size

Stress and strain are the properties that help us distinguish between the intrinsic properties of a material and those that are functions of its shape and size

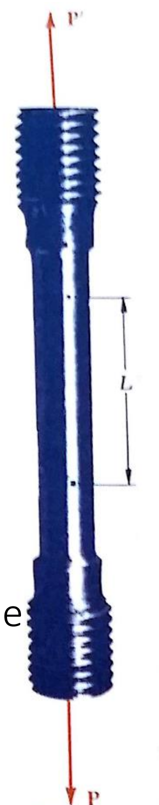
$$\sigma = \frac{F}{A} \qquad\qquad \varepsilon = \frac{\Delta l}{l_0}$$

Hooke was the first one to notice that the extension of a material is apparently proportional to the load applied to it
e.g. a steel wire stretches 1 cm under a load of 100 Newtons while it stretches 2cm under 200 Newtons

Young formulized the findings of Hooke in the form: $\frac{Stress}{Strain} = constant = E$

This theory which is the basis of *elasticity* is only an approximation that is valid for small deflections

The deformation of the steel rod is not only proportional to the load, but depends also on the intrinsic mechanical property ***E*** of the steel rod



In the physically stressful environment there are three ways in which a material can respond to external forces:

- It can add the load directly onto the forces that hold the constituent atoms or molecules together (the main mechanism in crystalline ceramics: enthalpic elastic response)
- It can feed the energy into large changes in shape (the main mechanism in non-crystalline polymers: entropic elastic response)
- It can flow away from the force to deform either semi-permanently (as with viscoelastic materials) or permanently (as with plastic materials)

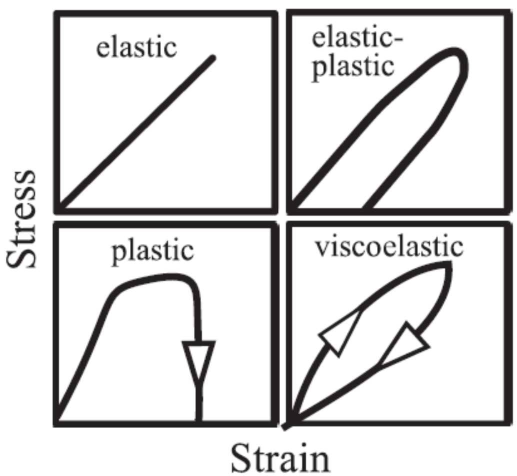


Figure 1.6. Stress–strain curves illustrating different types of behavior.

Elastic (Hookean) Materials

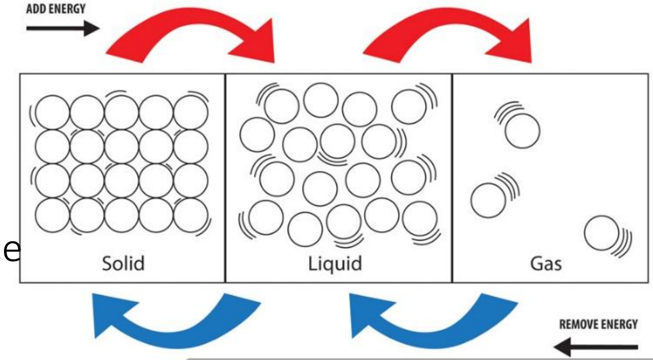
In a material at equilibrium, in the unloaded state, the distance between adjacent atoms is 0.1 to 0.2 nm.

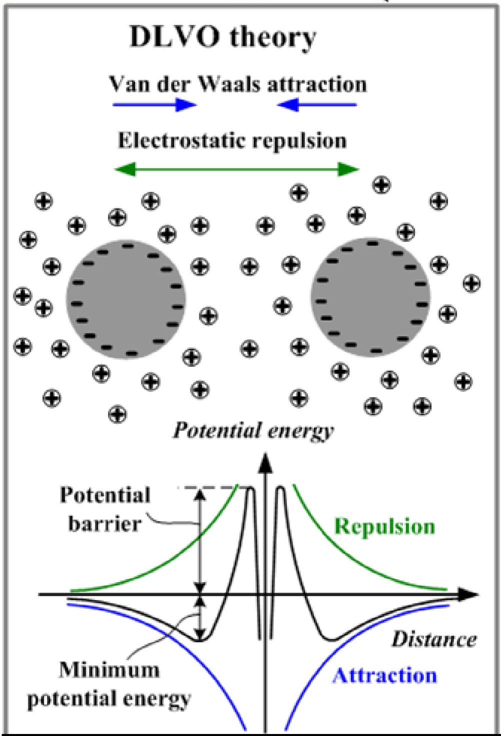
At this interatomic distance the forces of repulsion between two adjacent atoms balance the forces of attraction

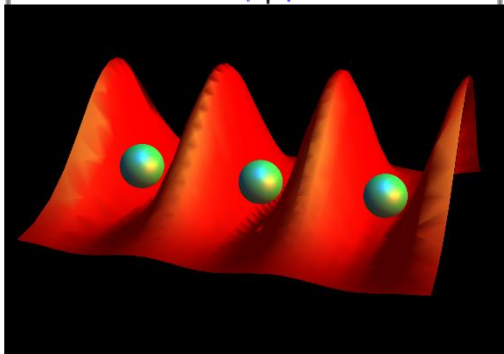
When the material is stretched or compressed, the atoms are forced out of their equilibrium positions

They are either parted or brought together until the forces generated between them, either of attraction or repulsion, respectively, balance the external force

With most stiff materials the extension or compression is limited by other factors to less than 10% of the bond length







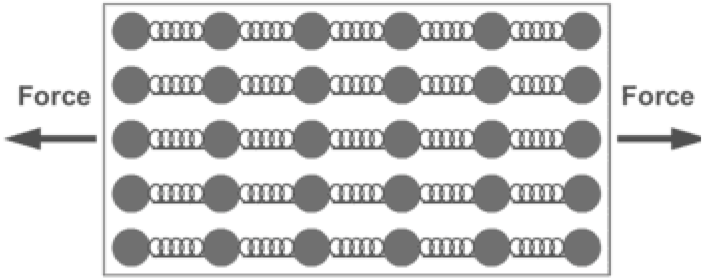
When the load is removed, the interatomic forces restore the atoms to their original equilibrium positions

A crystal consists of a large number of atoms held together by regular bonds.

The behavior of the entire crystal in response to the force is the summed responses of the individual bonds.

Hooke described this phenomenon as:
"*ut tensio, sic vis*" or "*as the extension, so the force*"

In other words for perfectly elastic materials the extension and force are directly and simply proportional to each other, and this relationship is a direct outcome of the behavior of the interatomic bond



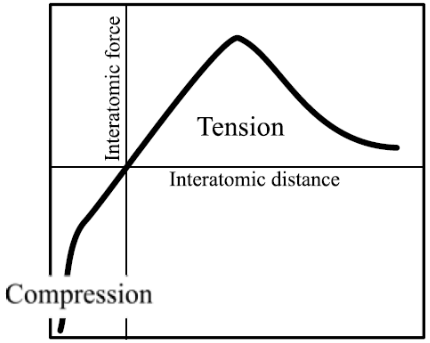
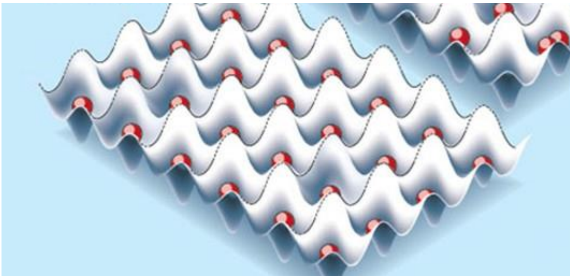


Figure 1.1. Stress–strain curve at the atomic level for a "perfect" material. The origin represents the equilibrium interatomic distance. On either side of the origin the curve is nearly straight.



The characteristic of Hookean materials is that they are perfectly elastic

They can be deformed within the elastic limit and will return to their original shape almost immediately after the force is removed (almost immediately because the stress wave travels through the material at the speed of sound in that material)

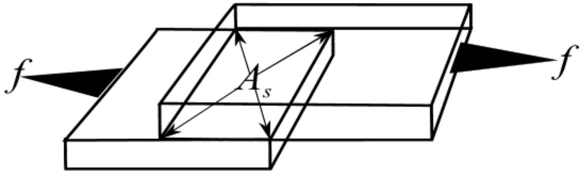
When we measure the response of the crystal to a stress, what is actually measured is the increase in length of the whole sample. We assume that the material is homogeneous and one part will deform as much as the whole to obtain the general relation between the stress and the strain of the material:

$$\sigma = E\varepsilon \qquad \text{Hooke's law}$$

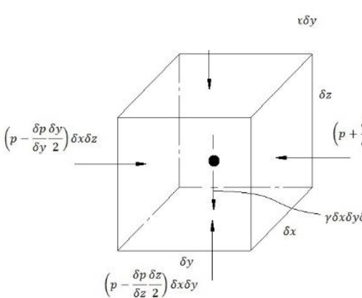
Young's modulus is a measure of stiffness in simple extension or compression

There are other types of deformation that have different effects on the interatomic forces and therefore different effects on the size and shape of the material

Shear stress $\tau = G\gamma$



Hydrostatic stress $\sigma_h = -ke$



Energy of deformation

When an elastic material is deformed, strain energy is stored in the deformation of its bonds

This energy brings the material back to (or close to) its original shape when the load is removed (energy can be dissipated or spent in a number of ways, such as heat, sound, surface energy, plastic deformation, or kinetic energy)

Total energy change in a material upon deformation: $A = U - TS$
where A is the Helmholtz free energy, ***U*** is the internal energy component, and ***–TS*** is the entropic component, made up of temperature, ***T***, and entropy, ***S***

Stains in a Hookean material are relatively small, and all the energy is stored in stretching the interatomic bonds, termed the internal energy, ***U = H-PV***

If the material is made of relatively long and unrestrained molecules, the energy can also be stored in changes in their shape and mobility, termed the entropic energy, ***S***

Consider the mechanical energy transfer through simple, uniaxial extension dl where the volume of the material is assumed constant (true for only perfectly plastic materials):
$E = F * dl$, therefore

$$F = \left(\frac{dU}{dl}\right)_{T,P} - T * \left(\frac{dS}{dl}\right)_{T,P} = \left(\frac{dH}{dl}\right)_{T,P} - P\left(\frac{dV}{dl}\right)_{T,P} - T * \left(\frac{dS}{dl}\right)_{T,P}$$

Consider also that simple extension results in some volume change dV (true for elastic materials)
$E = F * dl - P * dV$, therefore

So the applied load can be stored in two ways: enthalpy increase or entropy decrease

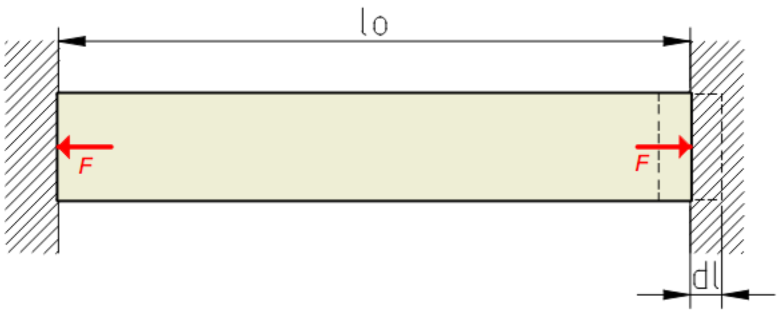
$$F = \left(\frac{dH}{dl}\right)_{T,P} - P\left(\frac{dV}{dl}\right)_{T,P} - T * \left(\frac{dS}{dl}\right)_{T,P} + P\left(\frac{dV}{dl}\right)_{T,P} = \underbrace{\left(\frac{dH}{dl}\right)_{T,P}}_{1} - T * \underbrace{\left(\frac{dS}{dl}\right)_{T,P}}_{2}$$

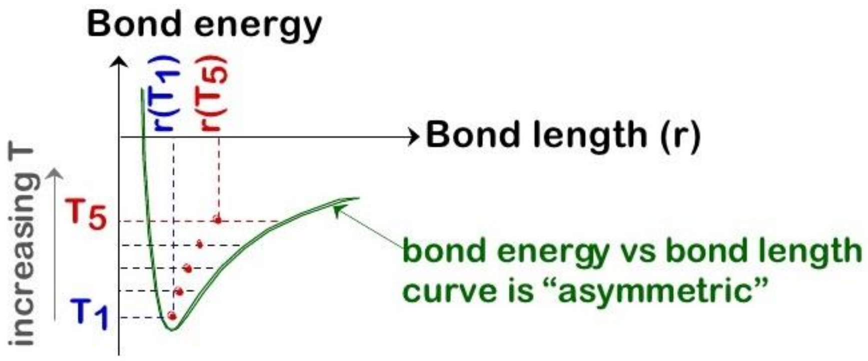
The first term dominates for a perfectly elastic crystalline material.
Stretching steel at small loads increases the average distance between the atoms in the lattice but there is not much ordering or disordering so dS is negligibly small

The second term dominates for elastomer rubber
Stretching a rubber band which consists of freely rotating molecular chains, distorts the chains from their most probable end-to-end positions to a less probable stretched state and decreases the entropy of the material. Because $\frac{dS}{dl}$ is negative, rubber will contract if you heat it!

Examples to energy change upon deformation

If you increase the temperature of a material that relies on enthalpy, ***H*** for its elastic behavior (like ceramic), it will expand:

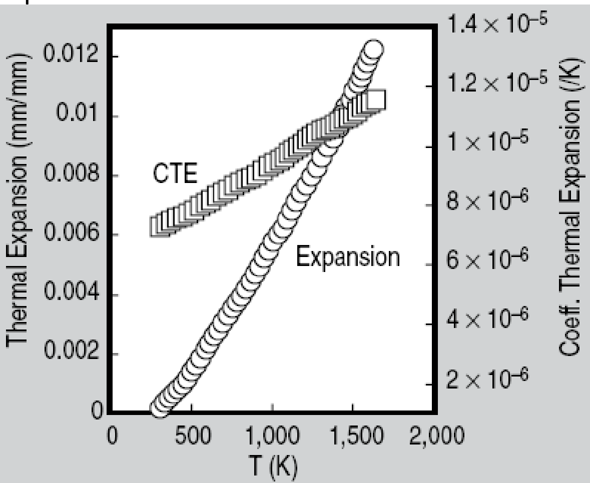




The asymmetric bond energy curve will result in a change in the average position of the atom

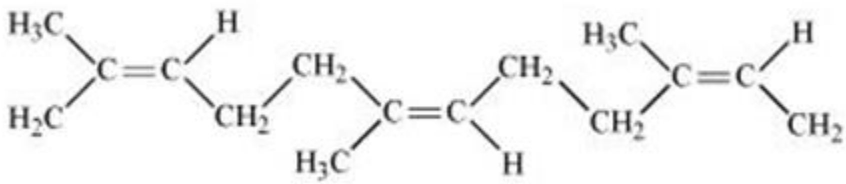
This causes an increase in inter-particle distance with increasing temperature

Thermal expansion coefficient also increases with temperature

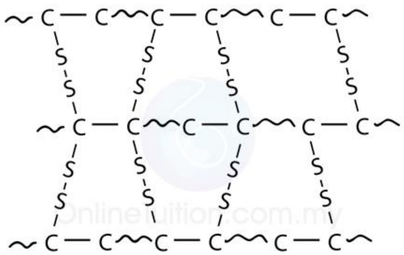


Examples to energy change upon deformation

An entropic energy based material (like rubber) will contract upon increase of temperature.

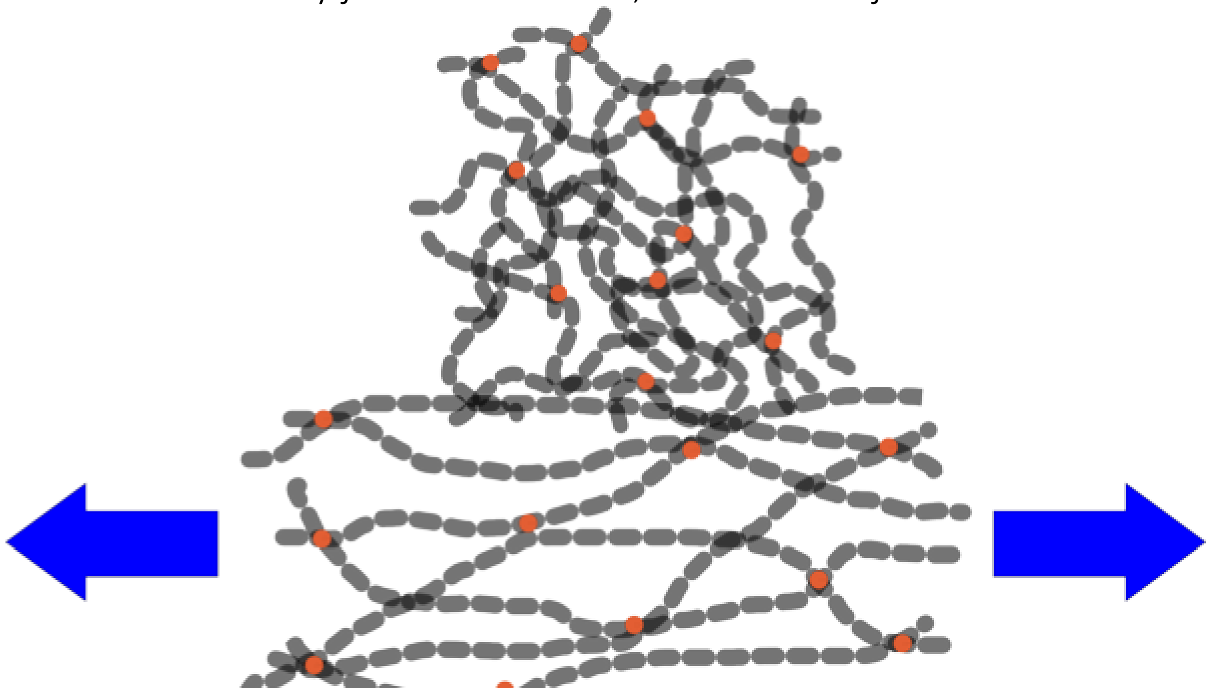


Natural rubber



Vulcanised rubber

A technical rubber is composed of very long chains (molecular weight of about 10^5) of one or more monomer units, with each unit more or less freely joint to the chain, so that each joint allows a wide range of movement.



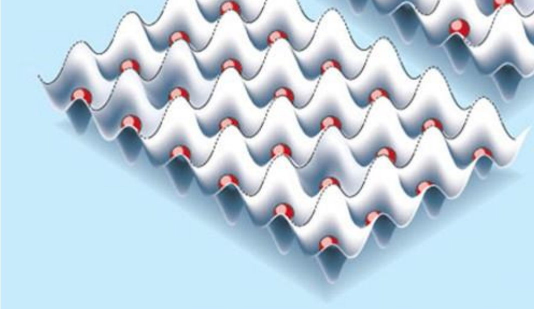
Elastomer response

The free rotation about the bonds of the backbone is what distinguishes a rubbery polymer from a crystalline one: in a crystalline polymer (or in areas of crystallinity) the units cannot move freely because they are packed so closely and have relatively low entropy.

As the temperature increases, the movement of the molecules and their subunits becomes more and more rapid.
Conversely, as the temperature decreases, the activity of the molecules slows until, finally, at a temperature dependent on the type of rubber, it ceases altogether and any force that is exerted on the rubber meets the resistance of the covalent bonds linking the atoms.

A rubber at the temperature of liquid nitrogen is Hookean and is glassy. The temperature at which this phenomenon occurs is called the **glass transition temperature**.

At normal temperatures the rubber chains rotate in Brownian motion. It is this rotation that produces the tension and elasticity of rubber. When we hold a rubber and stretch, we do work on the rotating molecules to decrease their entropy. As a result the rubber releases some heat. Rubber resists stretching more at high temperatures.



Molecular basis of Hooke’s law

Hooke’s law is an approximation even for a perfect crystalline solid

Consider pulling the perfect crystal in the figure from two ends uniaxially
The kinetic energy associated with the vibrations of the molecules around their average positions are negligible and the only contribution to internal energy is the potential energy associated with the chemical bonds that bind the molecules to each other.

On a line, each molecule is attached to two neighbors and each molecules is subject to potential energy that is the sum of two energy barriers. This energy curve can be represented by a function at small displacements:

$$PE = \frac{k}{2}x^2$$

Where x is the displacement from the minimum energy position which is considered as the origin of our one dimensional coordinate system, and k/2 is a constant.

At large displacements there are deviations from this function because of defects in microstructure:

$$PE = \frac{k}{2}x^2 + \frac{k'}{3}x^3 + \frac{k''}{4}x^4 + \cdots$$

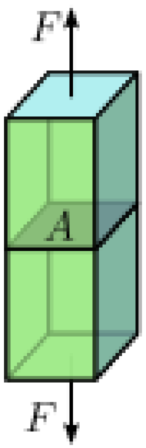
The simple approximation is true for small loads and displacements because $F = \dfrac{d(PE)}{dx}$

$$F = kx + k'x^2 + \cdots = k\Delta l + k'(\Delta l)^2 + \cdots$$

Higher terms can be neglected for small values of Δl, so $F = k\Delta l$

Elastic theory

Consider deformation of blocks under axial stresses



The normal stress is equal to the load per area of surface perpendicular to the load

$$\sigma = \frac{F}{A}$$

The deformation of the material per unit length is the normal strain, ε

$$\varepsilon = \frac{\delta}{L}$$

The stress-strain diagram helps us determine the modulus of elasticity of the material, whether it is ductile or brittle and whether the strains in the block will disappear when the load is removed

The stress-strain diagram of a material is obtained by conducting a tensile test on the specimen of material

The length and the cross-sectional area are recorded as the load is increased at a constant rate

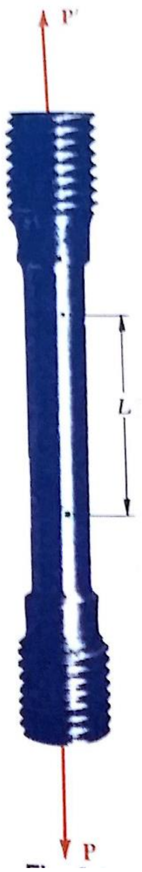
A material elongates linearly at a very slow rate as the load is increased according to the elastic theory

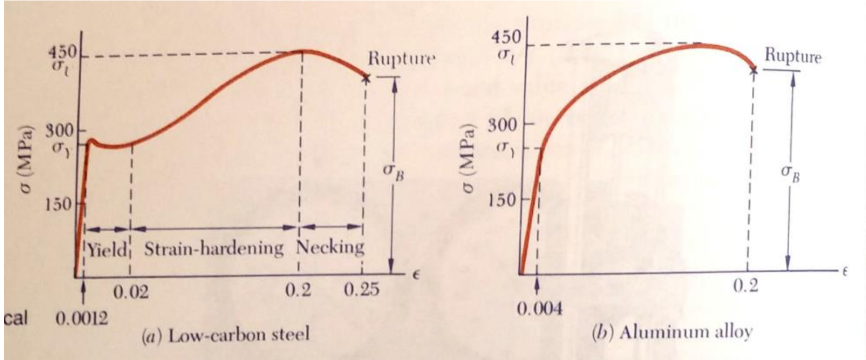
After yield stress is reached the specimen undergoes a large deformation with a relatively small increase in the applied load.

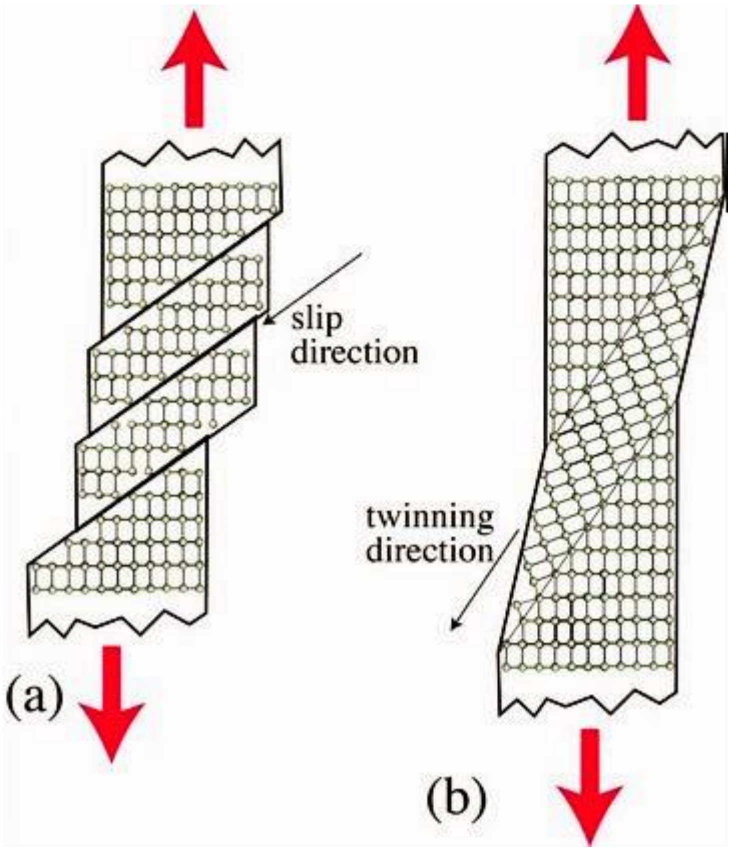
Yielding is caused by slippage of the material along oblique crystal surfaces and is primarily due to shearing stresses.

The diameter of a portion of the specimen begins to decrease due to local instability and results in necking

Lower loads are sufficient to rupture the specimen after necking
Rupture of ductile materials occurs along a cone-shaped surface which forms an angle of approximately 45 with the normal surface which shows that shear is primary cause of rupture.







Fracture surface of a stretched ductile material with both elastic and plastic deformation

Mechanisms of plastic deformation in metals

In axial stretching, value of σ obtained as $\frac{F}{A}$ represents the average stress over the section rather than the stress at a specific point P. Note that the stresses are uniform only when the direction of the load passes through the centroid of the material.

The stress at point P is defined as

$$\sigma = \lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A}$$

The stress value at point P is generally different than the average stress and is found to vary across the section. This variation is small in any section away from the points of application of the loads

In practice, the distribution of the normal stress in an axially loaded material is assumed to be uniform, except in the immediate vicinity of the points of application of the loads.

Consider the case when two rigid plates are used to transmit loads to the material
The plates move towards each other, causing the material to get shorter and increase in width and thickness, the distribution of stresses is uniform throughout the material and $\sigma_y = (\sigma_y)_{ave} = \frac{P}{A}$

Consider now that the load is concentrated
The material in the immediate vicinity of the points of application of the load are subjected to very large stresses
Other parts of the material away from the ends, at a distance equal to or greater than the width of the material are unaffected and load is distributed uniformly

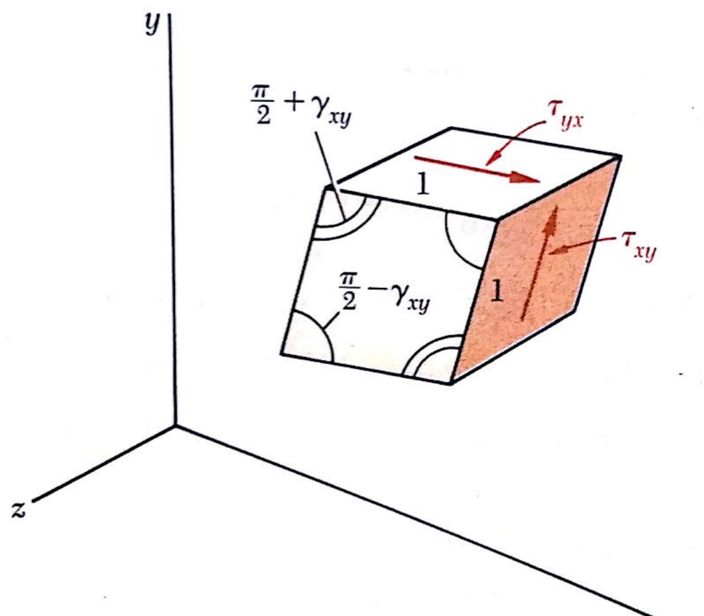
Shearing stresses are created when transverse forces are applied to a material.

These stresses vary greatly across the section and their distribution cannot be assumed uniform

The average shearing stress over the section is obtained by dividing the shearing load by the cross-sectional area:

$$\tau_{ave} = \frac{F}{A}$$

The deformation of a three dimensional structure under transverse loads is represented as the shear strain which is the change in angle of the initially perpendicular normal and transverse surfaces.



Both normal and shearing stresses develop in a material subjected to axial loading.

If we consider an oblique section inside the material under axial stress, the normal and shear stresses are related to the angle θ formed by the section with a

$$\sigma = \frac{F}{A_0}\cos^2\theta$$

$$\tau = \frac{F}{A_0}\sin\theta\cos\theta$$

where A_0 is the area of a section perpendicular to the axis of the material

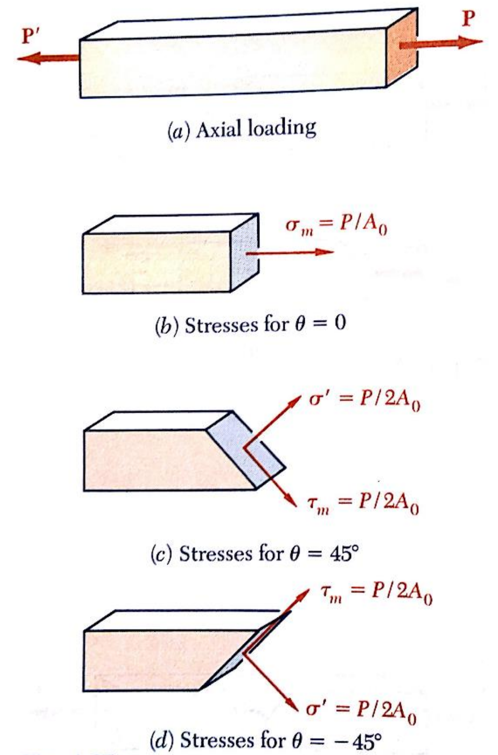


Fig. 1.27

Note that the normal stress is maximum and equal to $\frac{F}{A_0}$ for θ=0, while the shear stress is maximum and equal to $\frac{F}{2A_0}$ for θ=45

Most structural materials and machine components are under more complex loading conditions than axial and transverse loadings

For a point Q inside a three dimensional body subjected to various loads in various directions, the stress condition created by the loads can be determined.

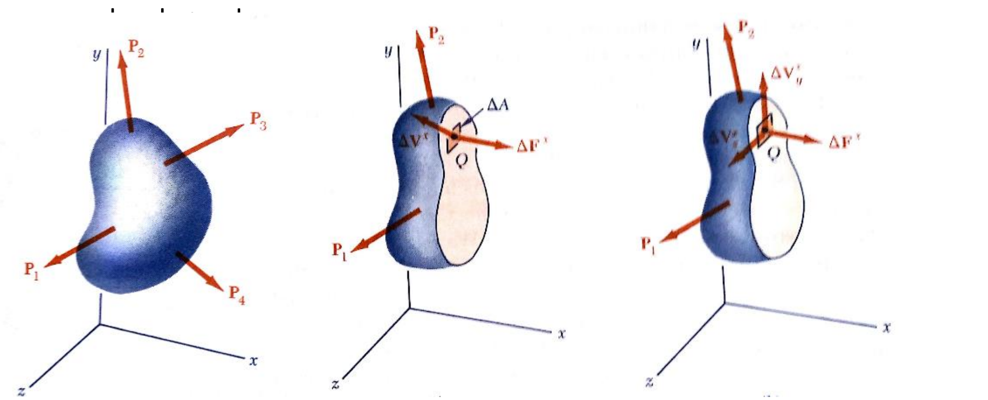
Sectioning the body at point Q using a plane parallel to the yz plane will help us visualize the stress conditions at the plane and point of Q.

The normal and shear stresses on point Q

$$\sigma_x = \lim_{\Delta A \to 0} \frac{\Delta F^x}{\Delta A}$$

$$\tau_{xy} = \lim_{\Delta A \to 0} \frac{\Delta V^x{}_y}{\Delta A}$$

$$\tau_{xz} = \lim_{\Delta A \to 0} \frac{\Delta V^x{}_z}{\Delta A}$$



where ΔF^x is the normal force, ΔV^x is the shearing force acting on the surface perpendicular to the x-axis, τ_{xy} is the shear stress perpendicular the x-axis, parallel to the y-axis and τ_{xz} is the shear stress perpendicular the x-axis, parallel to the z-axis.

While the normal force ΔF^x has a well-defined direction, the shearing force ΔV^x may have any direction in the plane of the section. So it is resolved into two component forces $\Delta V^x{}_y$, $\Delta V^x{}_z$ in direction parallel to the y and z axes.

When that analysis is applied to the portion of body located to the right of the vertical plane through Q, the same magnitudes but opposite directions are obtained for the normal and shearing forces ΔF^x,$\Delta V^x{}_y$, $\Delta V^x{}_z$

This time the section faces the negative x axis, so a positive sign for σ_x indicates that the corresponding arrow points in the negative x direction and negative y and z directions for τ_{xy}, τ_{xz}

The same analyses can be done parallel to the zx plane to obtain the stress components σ_y, τ_{yz}, τ_{yx} and to the xy plane to obtain components σ_z, τ_{zx}, τ_{zy}

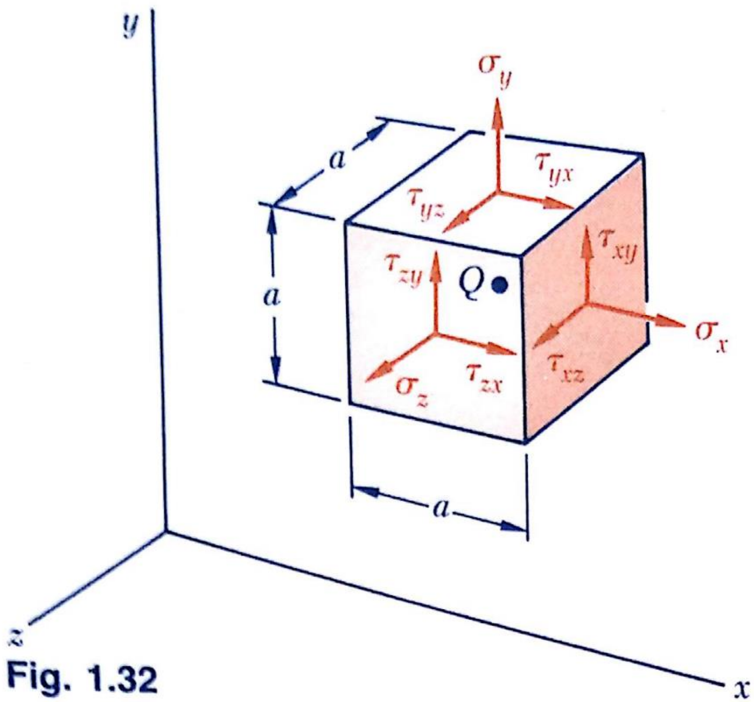


Fig. 1.32

Poisson’s ratio

The normal stresses acting on the three faces of an axially loaded specimen of material are

$$\sigma_x = E\epsilon_x, \sigma_y = 0, \sigma_z = 0$$

But the corresponding normal strains are

$$\epsilon_x = {}^{\sigma_x}/_{E}\,, \epsilon_y = \epsilon_z < 0$$

because an elongation produced by an axial tensile force is accompanied by a contraction in any transverse direction in all isotropic, homogeneous materials

The absolute value of the ratio of the lateral strain to the axial strain is the Poisson’s ratio

$$\upsilon = \left|\frac{\epsilon_y}{\epsilon_x}\right| = -\frac{\epsilon_y}{\epsilon_x} = -\frac{\epsilon_z}{\epsilon_x}$$

So

$$\epsilon_x = {}^{\sigma_x}/_{E}\,, \epsilon_y = \epsilon_z = -{}^{\sigma_x \upsilon}/_{E}$$

The volume of the material also changes as a result of axial elongation and transverse contraction

Multiaxial stress

The stress condition at point Q is understood clearly by considering a small cube of side a, centered at Q and the stresses exerted on each face of the cube.

Only three faces are visible but equal and opposite stress components act on the faces at the back.

As cube length ***a*** gets smaller, the error involved in the difference between stresses at the face and point is minimized. If we consider the point Q as a small cube we can calculate the normal and shearing forces acting on the various faces with area ΔA.

The condition for mechanical equilibrium for an object is that all of the forces and moments acting on three axes add up to zero:

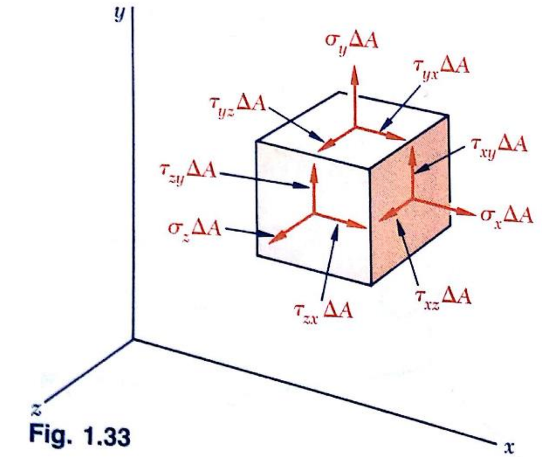


Fig. 1.33

$$\sum F_x = 0 \, , \sum F_y = 0 \, , \sum F_z = 0$$

$$\sum M_x = 0 \, , \sum M_y = 0 \, , \sum M_z = 0$$

$\sum F_{x,y,z} = 0$ are satisfied since equal and opposite forces are acting on parallel faces of the cube

$\sum M_z = 0$ is satisfied because the shear forces rotating the object around the z-axis are opposite, one causing counterclockwise and the other clockwise rotation:

$$\sum M_z = (\tau_{xy}\Delta A)\, a/2 - (\tau_{xy}\Delta A)\, a/2 = 0$$

And we conclude that

$$\tau_{xy} = \tau_{yx}$$

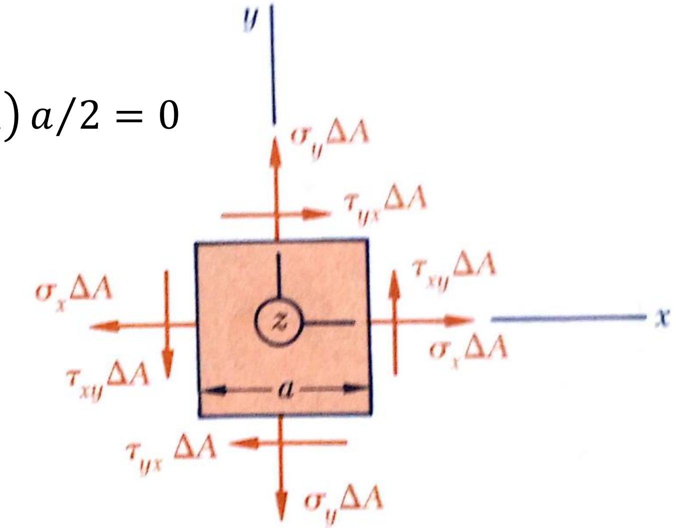


Fig. 1.34

Similarly from the remaining equations $\sum M_x = 0$, $\sum M_y = 0$, the relations between other shear stresses are obtained:

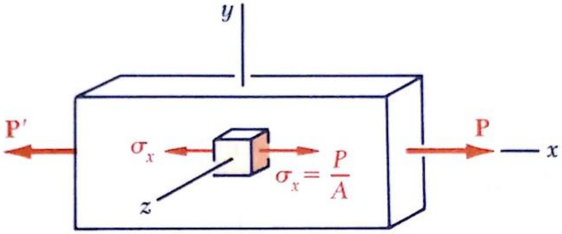
$$\tau_{yz} = \tau_{zy} \quad \tau_{zx} = \tau_{xz}$$

Hence it is seen that only 6 stress components are required to define the condition of stress at a given point Q

$$\sigma = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix}$$

Shear stresses are present in all stressed materials except hydrostatically stressed ones

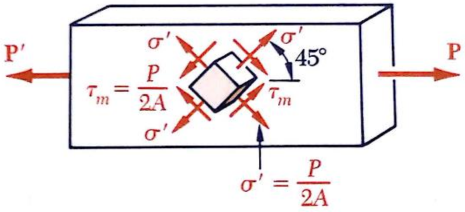
Consider the case of axial loading. The conditions of stress in part of the material is described as:



The only stresses acting on the small cube parallel to the loaded surface are normal stresses because

$$\sigma = \frac{P}{A_0}\cos^2 0 = \frac{P}{A_0}$$
$$\tau = \frac{P}{A_0}\sin 0 \cos 0 = 0$$

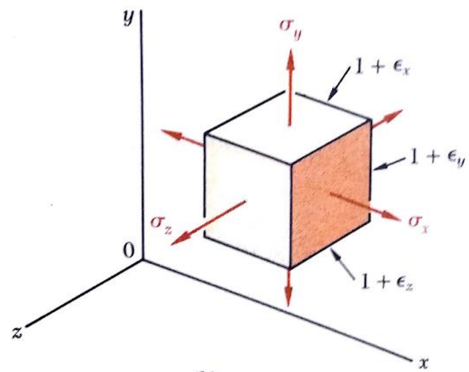
However if the small cube is rotated by 45 around z-axis, normal and shearing stresses of equal magnitude are exerted on four faces of the cube



The same loading condition may lead to different deformations at different points in a body, depending on the orientation of the element considered.

In the case of multiaxial loading of a material, deformations and the strains can be calculated by dividing the loads into normal and shearing forces in the three axis.

Consider the material as a unit cube with sides equal to 1. Consider also only the normal components of stress to calculate the normal strains as a generalized Hooke’s law



The effect of each normal stress component on the strain components ϵ_x, ϵ_y, ϵ_z should be calculated separately and combined.

$$\epsilon_x = +\frac{\sigma_x}{E} - \frac{\nu\sigma_y}{E} - \frac{\nu\sigma_z}{E}$$

$$\epsilon_y = -\frac{\nu\sigma_x}{E} + \frac{\sigma_y}{E} - \frac{\nu\sigma_z}{E}$$

$$\epsilon_z = -\frac{\nu\sigma_x}{E} - \frac{\nu\sigma_y}{E} + \frac{\sigma_z}{E}$$

This result is valid only as long the stresses do not exceed the proportional limit and as long as the deformations involved are small

To consider the shear strains in a real stress situation, the shearing stress components should also be considered

Shearing stresses tend to deform a cubic material into an oblique parallelepiped

Two of the angles formed by the four faces under stress are reduced from $\frac{\pi}{2}$ to $\frac{\pi}{2} - \gamma_{xy}$ and two angles are increased from $\frac{\pi}{2}$ to $\frac{\pi}{2} + \gamma_{xy}$

The angle γ_{xy} is the shear strain corresponding to the x and y directions in the plane of z

Shear strains on other planes may also be calculated similarly

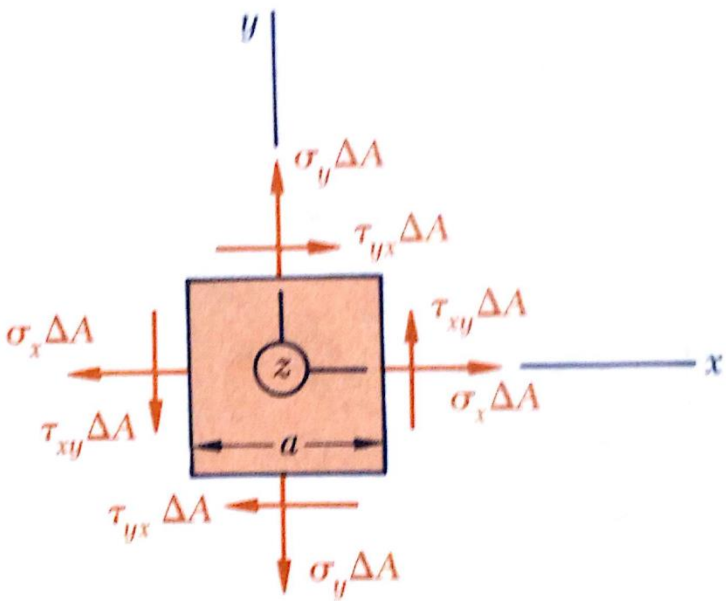
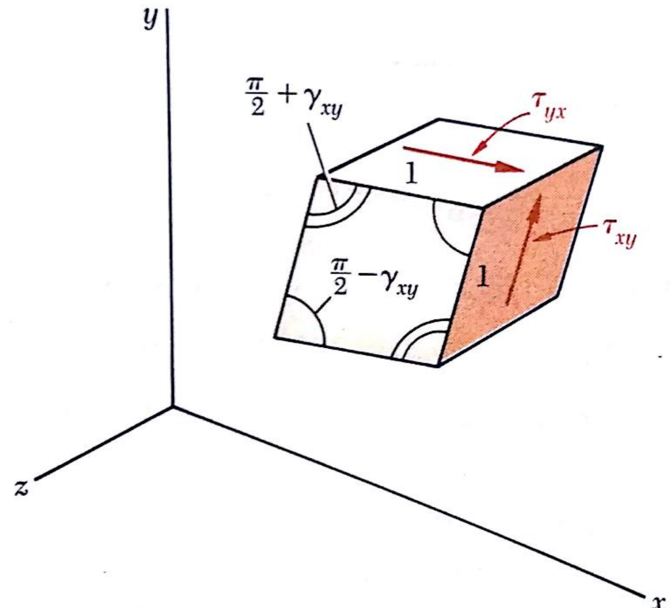


Fig. 1.34



Plotting successive values of τ_{xy} against γ_{xy} gives us the shearing stress-strain diagram for the material

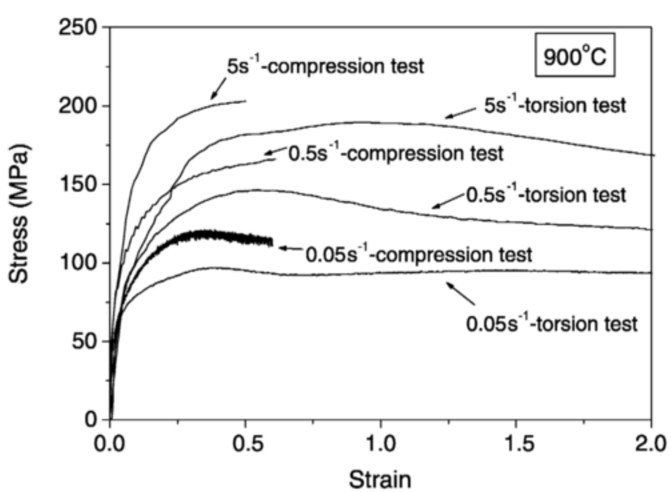
Obtained yield strength, ultimate strength values are about half of those from the tensile test

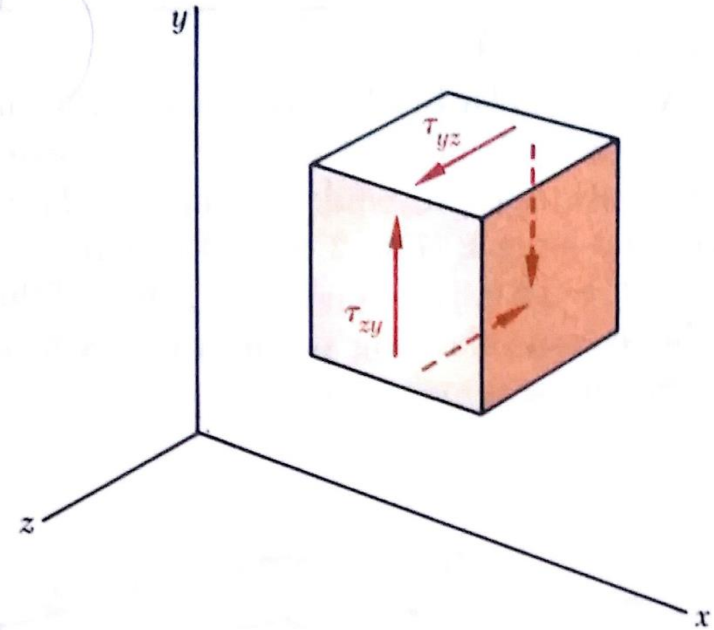
For values of the shearing stress which do not exceed the proportional limit in shear, Hooke's law for shearing is defined as

$$\tau_{xy} = G\gamma_{xy}$$

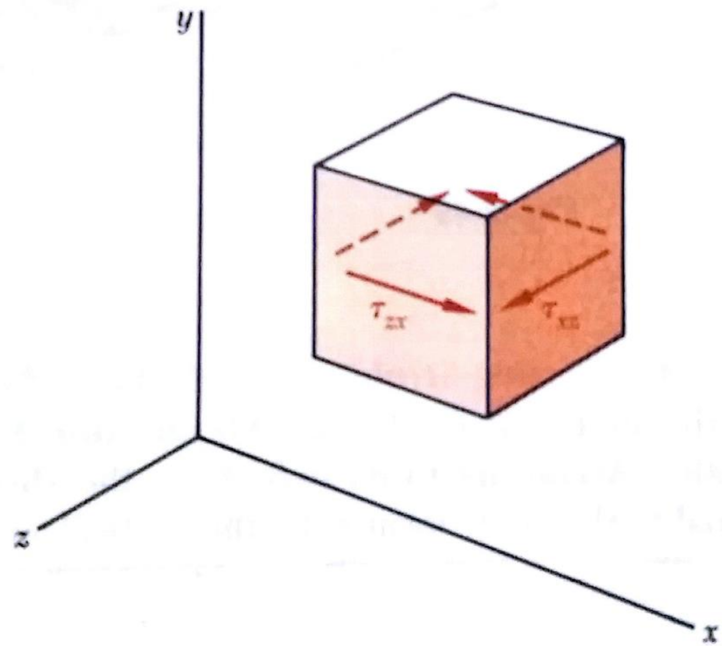
The relations for the other planes are obtained by considering the corresponding stress components

$$\tau_{yz} = G\gamma_{yz} \qquad \tau_{zx} = G\gamma_{zx}$$





(a)



The following group of equations representing the generalized Hooke's law for a homogeneous isotropic material under multiaxial loading is obtained:

$$\epsilon_x = +\frac{\sigma_x}{E} - \frac{\nu\sigma_y}{E} - \frac{\nu\sigma_z}{E}$$

$$\epsilon_y = -\frac{\nu\sigma_x}{E} + \frac{\sigma_y}{E} - \frac{\nu\sigma_z}{E}$$

$$\epsilon_z = -\frac{\nu\sigma_x}{E} - \frac{\nu\sigma_y}{E} + \frac{\sigma_z}{E}$$

$$\frac{\tau_{xy}}{G} = \gamma_{xy} \qquad \frac{\tau_{yz}}{G} = \gamma_{yz} \qquad \frac{\tau_{zx}}{G} = \gamma_{zx}$$

The 6 strain components of a material under stress in the elastic region can be calculated from the 6 stress components and any 2 of E, G, ν

$$\epsilon = \begin{bmatrix} +\frac{\sigma_x}{E} - \frac{\nu\sigma_y}{E} - \frac{\nu\sigma_z}{E} & \frac{\tau_{xy}}{G} & \frac{\tau_{xz}}{G} \\ \frac{\tau_{xy}}{G} & -\frac{\nu\sigma_x}{E} + \frac{\sigma_y}{E} - \frac{\nu\sigma_z}{E} & \frac{\tau_{yz}}{G} \\ \frac{\tau_{xz}}{G} & \frac{\tau_{yz}}{G} & -\frac{\nu\sigma_x}{E} - \frac{\nu\sigma_y}{E} + \frac{\sigma_z}{E} \end{bmatrix}$$

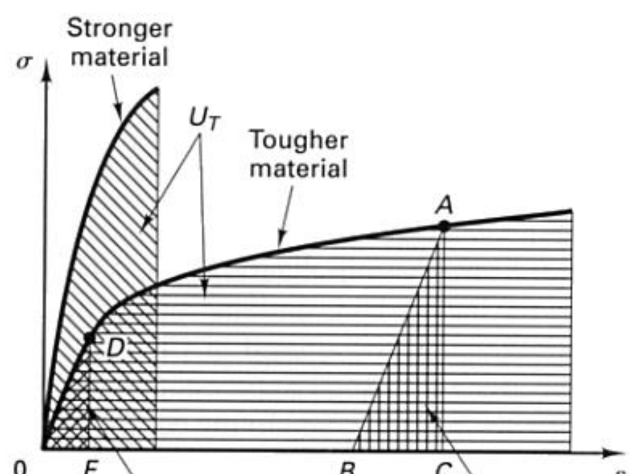
The strain energy of deformation per unit of a linear elastic material is

$$U_0 = \frac{1}{2}\sum \sigma_{ij}\epsilon_{ij}$$

$$U_0 = \frac{1}{2}\left(\sigma_{xx}\epsilon_{xx} + \sigma_{yy}\epsilon_{yy} + \sigma_{zz}\epsilon_{zz} + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{zx}\gamma_{zx}\right)$$

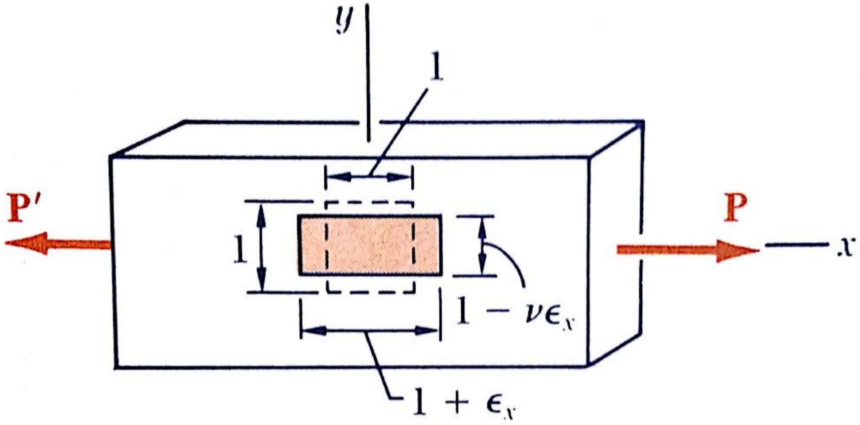
$$\sigma = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix} \qquad \epsilon = \begin{bmatrix} +\frac{\sigma_x}{E} - \frac{\nu\sigma_y}{E} - \frac{\nu\sigma_z}{E} & \frac{\tau_{xy}}{G} & \frac{\tau_{xz}}{G} \\ \frac{\tau_{xy}}{G} & -\frac{\nu\sigma_x}{E} + \frac{\sigma_y}{E} - \frac{\nu\sigma_z}{E} & \frac{\tau_{yz}}{G} \\ \frac{\tau_{xz}}{G} & \frac{\tau_{yz}}{G} & -\frac{\nu\sigma_x}{E} - \frac{\nu\sigma_y}{E} + \frac{\sigma_z}{E} \end{bmatrix}$$

$$U_0 = \frac{1}{2}\left(\sigma_x\left(\frac{\sigma_x}{E} - \frac{\nu\sigma_y}{E} - \frac{\nu\sigma_z}{E}\right) + \sigma_y\left(-\frac{\nu\sigma_x}{E} + \frac{\sigma_y}{E} - \frac{\nu\sigma_z}{E}\right) + \sigma_z\left(-\frac{\nu\sigma_x}{E} - \frac{\nu\sigma_y}{E} + \frac{\sigma_z}{E}\right) + \tau_{xy}\frac{\tau_{xy}}{G} + \tau_{yz}\frac{\tau_{yz}}{G} + \tau_{xz}\frac{\tau_{xz}}{G}\right)$$



The relationship between Poisson's ratio, Elastic and Shear moduli

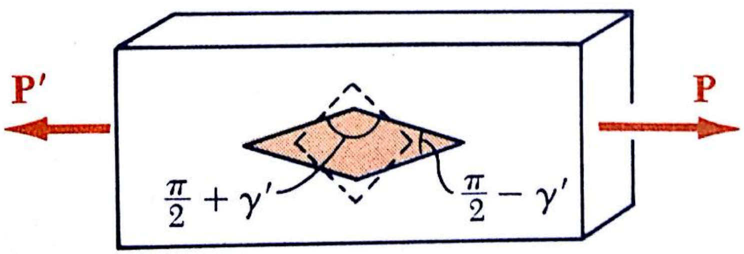
A material of rectangular prism geometry that is subjected to axial loading deforms into a rectangular parallelepiped. The prism will elongate along the axis of the tensile force and will contract in both of the transverse y and z directions:



(a)

For an element in the shape of a cube with unit sides, the deformed geometry will have sides $1+\epsilon_x$, $1-\nu\epsilon_x$, $1-\nu\epsilon_x$ according to the Hooke's law

If the element is oriented at 45 to the axis of the load, the cube will transform into an oblique parallelepiped due to the shearing stress components:



(b)

Each of the right angles increases or decreases by the shearing strain γ'that is induced by the shear stress component of the axial stress and $\gamma' = \gamma_{max}$

Consider the prismatic element obtained by intersecting the unit cubic element by a diagonal plane

It deforms into a slice which has horizontal and vertical sides equal to $1+\epsilon_x$ and $1-\nu\epsilon_x$ by the applied force.
The angles of the undeformed and the deformed slices are one half of $\frac{\pi}{2}$ and $\frac{\pi}{2}-\gamma_{max}$ respectively

$$\beta = \frac{\pi}{4} - \frac{\gamma_{max}}{2}$$

$$\tan\beta = \frac{\tan\frac{\pi}{4} - \tan\frac{\gamma_{max}}{2}}{1+\tan\frac{\pi}{4}\tan\frac{\gamma_{max}}{2}} = \frac{1-\tan\frac{\gamma_{max}}{2}}{1+\tan\frac{\gamma_{max}}{2}}$$

Since $\frac{\gamma_{max}}{2}$ is a very small angle,

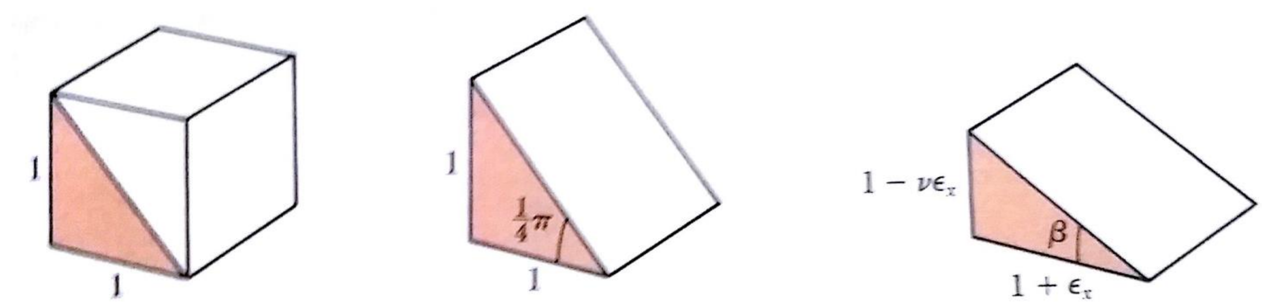
$$\tan\beta = \frac{1-\frac{\gamma_{max}}{2}}{1+\frac{\gamma_{max}}{2}}$$

Also

$$\tan\beta = \frac{1-\nu\epsilon_x}{1+\epsilon_x}$$

Hence,

$$\gamma_{max} = \frac{(1+\nu)\epsilon_x}{1+\frac{1-\nu}{2}\epsilon_x}$$



$$\gamma_{max} = \frac{(1+\nu)\epsilon_x}{1 + \frac{1-\nu}{2}\epsilon_x}$$

The denominator may be taken as 1 since $\epsilon_x \ll 1$, and the relation between the maximum shearing strain and the axial strain is obtained:

$$\gamma_{max} = (1+\nu)\epsilon_x$$

According to Hooke's law,

$$\gamma_{max} = \frac{\tau_{max}}{G} \qquad \epsilon_x = \frac{\sigma_x}{E}$$

$$\frac{\tau_{max}}{G} = (1+\nu)\frac{\sigma_x}{E}$$

$$\frac{E}{G} = (1+\nu)\frac{\sigma_x}{\tau_{max}}$$

Recall that the axial stress and the maximum shearing stress are defined as $\sigma_x = \frac{P}{A}$ and $\tau_{max} = \frac{P}{2A}$

$$\frac{E}{G} = 2(1+\nu)$$

or

$$\frac{E}{2G} = (1+\nu)$$

Hence the constants E, G and ν are related to each other for any material