Plasticity and Deformation Process

Stress-strain relations in deformation theory

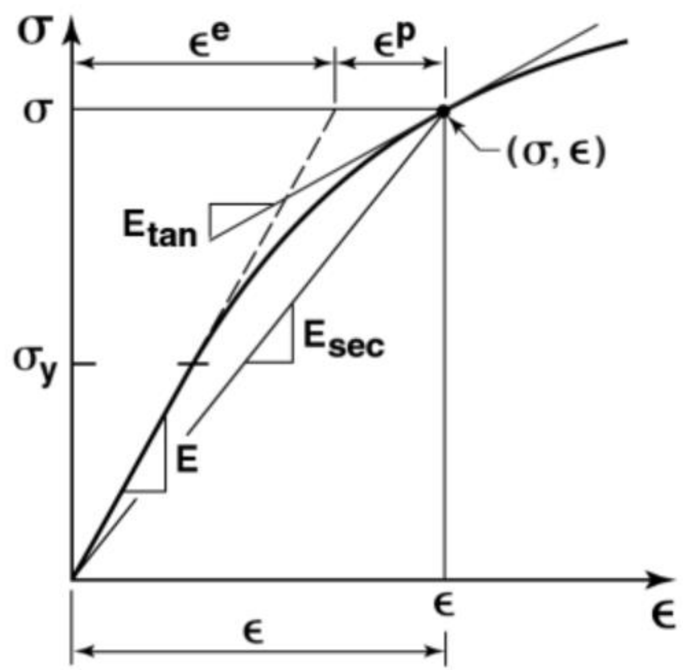
The fundamental problem in the solution of a plasticity problem is to determine how stresses and strains can be found for a specified state of loading on a body.

There are two theories to describe the relation between stresses and strains:

Deformation or total strain theory:

Total strains are directly related to the total stresses by the secant modulus which is a function of the stress level
The strains on an object depend on the final state of stress, they are independent of stress history

Flow or incremental strain theory:
Increments of plastic strain $\Delta\varepsilon^P$ are related to increments of plastic stress $\Delta\sigma^P$ by the tangent modulus which is a function of the stress level

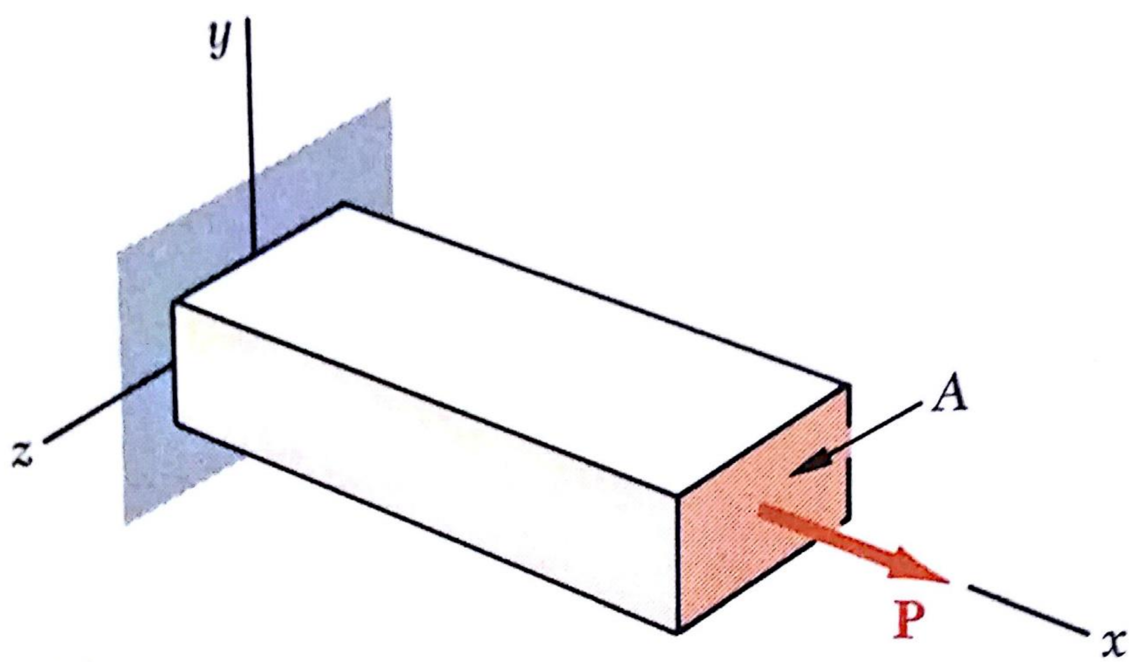


The strains are separated into an elastic component ε^e and a plastic component ε^P in both theories.

The elastic component of strain under uniaxial stress loading is σ/E

The plastic component is represented by a complex equation as will be derived next

Both deformation and incremental theories are based on the assumption that elastic deformation is compressible and plastic deformation is incompressible

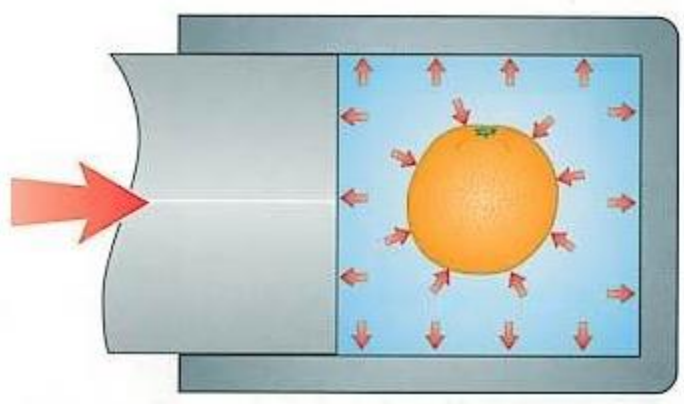


(a)

The compressible nature of elastic deformation is obvious from the fact that Poisson's ratio for ordinary isotropic materials is much less than one-half (0-0.35)

The incompressibilty of plastic deformation is not obvious

Experiments on materials subjected to very high hydrostatic pressures show that the density and volume do not significantly change under extremely high pressures.



Recall the dilatation or volumetric strain of materials under hydrostatic stress:

$$\theta = \varepsilon_x + \varepsilon_y + \varepsilon_z = \varepsilon(1 - 2\nu)$$

In which ε is the total strain under uniaxial stress σ and ν is the Poisson's ratio for total strains

Dilatation can be regarded as the sum of an elastic dilatation and a plastic dilatation

$$\theta^e = \varepsilon^e(1 - 2\nu^e) \qquad \theta^p = \varepsilon^p(1 - 2\nu^p)$$

$$\theta = \theta^e + \theta^p = \varepsilon(1-2\nu) = \varepsilon^e(1-2\nu^e) + \varepsilon^p(1-2\nu^p)$$

Dividing both sides by ε and substituting $\varepsilon - \varepsilon^e$ for ε^p gives:

$$(1-2\nu) = \frac{\varepsilon^e}{\varepsilon}(1-2\nu^e) + \frac{\varepsilon - \varepsilon^e}{\varepsilon}(1-2\nu^p)$$

$$-2\nu = \frac{\varepsilon^e}{\varepsilon}(-2\nu^e) - 2\nu^p + \frac{\varepsilon^e}{\varepsilon}2\nu^p$$

So

$$\nu = \nu^p - \frac{\varepsilon^e}{\varepsilon}(\nu^p - \nu^e)$$

Furthermore the strains can be expressed in terms of the moduli:

$$\frac{\varepsilon^e}{\varepsilon} = \frac{\sigma/E}{\sigma/E_{sec}} = \frac{E_{sec}}{E}$$

Finally

$$\nu = \nu^p - \frac{E_{sec}}{E}(\nu^p - \nu^e)$$

Or

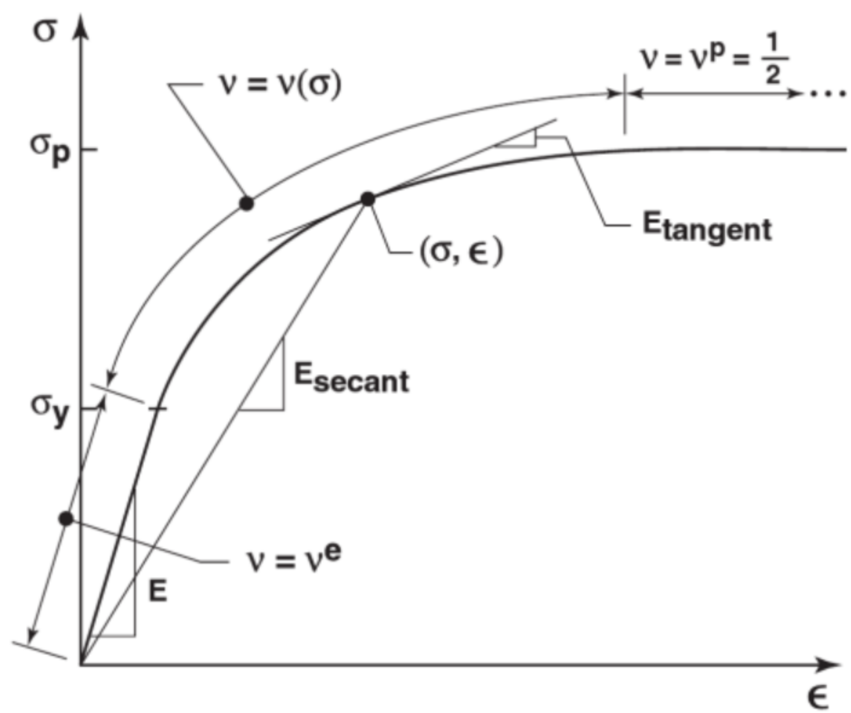
$$\nu = \frac{1}{2} - \frac{E_{sec}}{E}\left(\frac{1}{2} - \nu^e\right)$$

Since materials are incompressible during plastic deformation, the Poisson's ratio changes from the elastic value to the incompressible value ½ in a very gradual way as the stress is increased above the yield stress

$$\nu = \nu^e \qquad \text{for } \sigma \le \sigma_y$$

$$\nu = \nu(\sigma) = \frac{1}{2} - \frac{E_{sec}}{E}\left(\frac{1}{2} - \nu^e\right) \qquad \text{for } \sigma > \sigma_y$$

$$\nu = \frac{1}{2} \qquad \text{for } \sigma \gg \sigma_y$$



The stress-strain relations are expressed by Hencky et al. according to the deformation theory

Strains are separated into elastic and plastic strains:

$$\varepsilon_x = {\varepsilon_x}^e + {\varepsilon_x}^p$$

$$\gamma_{xy} = {\gamma_{xy}}^e + {\gamma_{xy}}^p$$

The elastic strains are obtained from the Hooke's law

$${\varepsilon_x}^e = \frac{1}{E}\left(\sigma_x - \nu^e(\sigma_y + \sigma_z)\right)$$

$${\varepsilon_y}^e = \frac{1}{E}\left(\sigma_y - \nu^e(\sigma_x + \sigma_z)\right)$$

$${\varepsilon_z}^e = \frac{1}{E}\left(\sigma_z - \nu^e(\sigma_x + \sigma_y)\right)$$

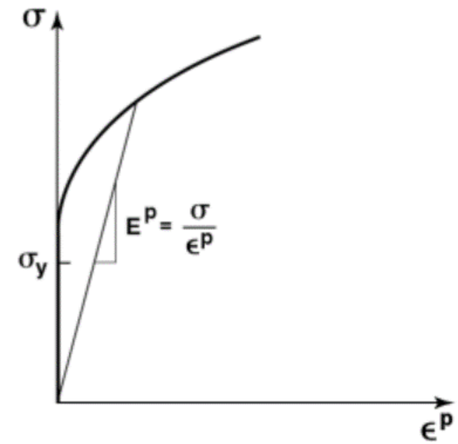
$${\gamma_{xy}}^e = \frac{\tau_{xy}}{G}$$

$${\gamma_{yz}}^e = \frac{\tau_{yz}}{G}$$

$${\gamma_{zx}}^e = \frac{\tau_{zx}}{G}$$

Where the mechanical properties E and G are the normal elastic modulus and shear modulus

To obtain the plastic moduli we need to consider the stress-strain diagram in terms of the normal stress and the plastic strain



The plastic modulus at any stress above the yield stress is the secant modulus at that point

$$\varepsilon_x{}^p = \frac{1}{E^p}\left(\sigma_x - \nu^p\left(\sigma_y + \sigma_z\right)\right)$$
$$\varepsilon_y{}^p = \frac{1}{E^p}\left(\sigma_y - \nu^p\left(\sigma_x + \sigma_z\right)\right)$$
$$\varepsilon_z{}^p = \frac{1}{E^p}\left(\sigma_z - \nu^p\left(\sigma_x + \sigma_y\right)\right)$$
$$\gamma_{xy}{}^p = \frac{\tau_{xy}}{G^p}$$
$$\gamma_{yz}{}^p = \frac{\tau_{yz}}{G^p}$$
$$\gamma_{zx}{}^p = \frac{\tau_{zx}}{G^p}$$

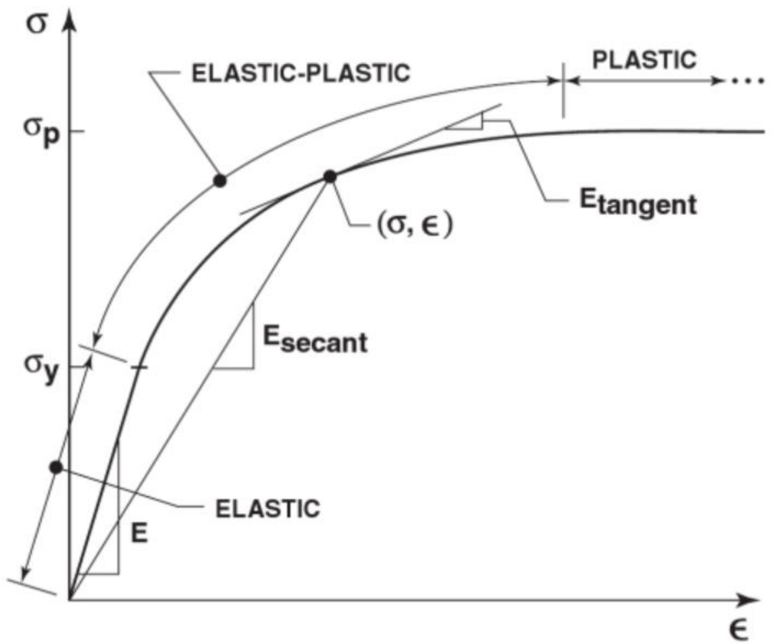
Where ν^p is the plastic Poisson's ratio (1/2) and G^p is the plastic shear modulus:

$$G^p = \frac{E^p}{2(1+\nu)} = \frac{E^p}{3}$$

The elastic and plastic strains sum to the total strains as

$$\frac{1}{E_{sec}} = \frac{1}{E} + \frac{1}{E^p}$$

$$\nu = \frac{\nu^e/E + \nu^p/E^p}{1/E + 1/E^p}$$



The stress-strain behavior is divided into three regions: elastic, elastic-plastic and plastic

Only the elastic-plastic region is considered in the deformation theory as a material nonlinearity

The modulus is expressed as a function of the stresses using various material models.

The problem-solving is not straight forward because the secant modulus depends on the stress and an iteration procedure is essential.

Hencky used the von Mises yield criterion and the distortional energy concept to derive the stress-strain relations

The combination of the yield criterion, the stress-strain relations, and the material model is the complete deformation theory of plasticity

Deformation theory helps us predict the stresses and strains at a point on the stress-strain curve but does not enable consideration of the path taken to get there.

Because of that, loading and unloading can not be evaluated with the same material model using deformation theory and should be considered as separate events.

Another limitation of deformation theory is that all stresses in a multiaxial stress state must be applied in proportion to one another because deformation theory is not capable of distinguishing between types of loading.

Also hardening is considered isotropic because the secant modulus is used instead of the tangent modulus.

These restrictions are valid for some plasticity problems and the theory is not generally applicable.

But it is applicable to most practical problems in metal forming and quite useful. Some problems that are easily solved with deformation theory are difficult to solve with incremental theory because of the excessively complex computation methods.

The basis for the deformation theory of plasticity is the stress-strain relations and the associated stress and strain intensities for multiaxial stress states.

The stress intensity or the effective stress for an elastic material is expressed as

$$\sigma_{eff} = \frac{\sqrt{2}}{2}\sqrt{\left(\sigma_x - \sigma_y\right)^2 + \left(\sigma_y - \sigma_z\right)^2 + (\sigma_z - \sigma_x)^2 + 6\left(\tau_{yz}{}^2 + \tau_{zx}{}^2 + \tau_{xy}{}^2\right)}$$

And the effective strain as

$$\varepsilon_{eff} = \frac{\sqrt{2}}{2(1+\nu)}\sqrt{\left(\varepsilon_x - \varepsilon_y\right)^2 + \left(\varepsilon_y - \varepsilon_z\right)^2 + (\varepsilon_z - \varepsilon_x)^2 + \frac{3}{2}\left(\gamma_{yz}{}^2 + \gamma_{zx}{}^2 + \gamma_{xy}{}^2\right)}$$

And

$$\sigma_{eff} = E\varepsilon_{eff}$$

The three dimensional elastic stress-strain relations for an isotropic material in terms of Young's modulus, and Poisson's ratio are:

$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_x + \nu\left(\varepsilon_y + \varepsilon_z\right)\right]$$

$$\sigma_y = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_y + \nu(\varepsilon_z + \varepsilon_x)\right]$$

$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_z + \nu\left(\varepsilon_x + \varepsilon_y\right)\right]$$

$$\tau_{yz} = \frac{E}{2(1+\nu)}\gamma_{yz}$$

$$\tau_{zx} = \frac{E}{2(1+\nu)}\gamma_{zx}$$

$$\tau_{xy} = \frac{E}{2(1+\nu)}\gamma_{xy}$$

Partitioning of the strains into elastic and plastic components helps us understand the deformation theory. In practice it is more convenient to determine the strains from a state of multiaxial stress in a single form that applies to all three deformation regions

These three dimensional total stress-strain relatios for an isotropic material in terms of Young's secant modulus and a continuously variable Poisson's ratio are similarly:

$$\sigma_x = \frac{E_{secant}}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_x + \nu\left(\varepsilon_y + \varepsilon_z\right)\right]$$

$$\sigma_y = \frac{E_{secant}}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_y + \nu(\varepsilon_z + \varepsilon_x)\right]$$

$$\sigma_x = \frac{E_{secant}}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_z + \nu\left(\varepsilon_x + \varepsilon_y\right)\right]$$

$$\tau_{yz} = \frac{E_{secant}}{2(1+\nu)}\gamma_{yz} = G_{secant}\gamma_{yz}$$

$$\tau_{zx} = \frac{E_{secant}}{2(1+\nu)}\gamma_{zx} = G_{secant}\gamma_{zx}$$

$$\tau_{xy} = \frac{E_{secant}}{2(1+\nu)}\gamma_{xy} = G_{secant}\gamma_{xy}$$

Where

$$\nu = \frac{1}{2} - \frac{E_{sec}}{E}\left(\frac{1}{2} - \nu^e\right)$$

And

$$G_{secant} = \frac{E_{sec}}{2(1+\nu)}$$

Most deformation processes involving thin plates of material are approximated to the plane stress conditions

Plane stress is a state of stress in which the normal stress σ_z, and the shear stresses σ_{xz}, σ_{yz} directed perpendicular to the x-y plane are assumed to be zero

The geometry of the body is that of a plate with one dimension much smaller than the others. The loads are applied uniformly over the thickness of the plate and act in the plane of the plate as shown.

$$\{\sigma\} = [D]\ \{\varepsilon\}$$

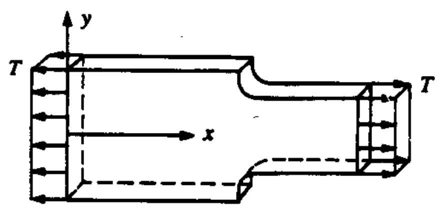


plate with fillet

$$[D] = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

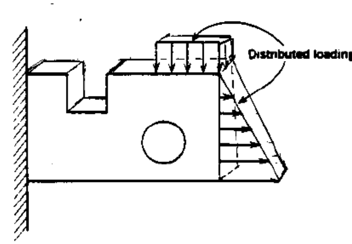


plate with hole

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix}\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix}$$

The plane stress condition is the simplest form of behavior for continuum structures and represents situations frequently encountered in practice

Under plane stress conditions $\sigma_z = 0$, $\tau_{xz} = \tau_{yz} = 0$ and $\varepsilon_z = -\frac{\nu}{1-\nu}\left(\varepsilon_x + \varepsilon_y\right)$

So the effective strain for a state of plane stress is

$$\varepsilon_{eff} = \frac{\sqrt{2}}{2(1-\nu^2)}\sqrt{(1-\nu+\nu^2)\left(\varepsilon_x{}^2 + \varepsilon_y{}^2\right) + (1-4\nu+\nu^2)\varepsilon_x\varepsilon_y + \frac{3}{4}(1-\nu)^2\gamma_{xy}{}^2}$$

And the effective stress is

$$\sigma_{eff} = \frac{\sqrt{2}}{2}\sqrt{\left(\sigma_x - \sigma_y\right)^2 + \left(\sigma_y\right)^2 + (\sigma_z)^2 + 6\left(\tau_{xy}{}^2\right)}$$

Example – A thin disk of aluminum is stressed under plane stress conditions. Determine the effective stress and strain resulting from the load if the normal stresses in the x and y direction are 70 MPa and 40 MPa, and the shearing stress on the plane is 30 MPa. E_{Al}= 70 GPa

Deformation theory is workable only if E_{sec} and ν are expressed as a function of the multiaxial stress state
Recall that

$$\nu = \frac{1}{2} - \frac{E_{sec}}{E}\left(\frac{1}{2} - \nu^e\right)$$

The E_{sec} should be expressed as a function of the multiaxial stress state

The general effective stress equation is

$$\sigma_{eff} = \frac{\sqrt{2}}{2}\sqrt{\left(\sigma_x - \sigma_y\right)^2 + \left(\sigma_y - \sigma_z\right)^2 + (\sigma_z - \sigma_x)^2 + 6\left({\tau_{yz}}^2 + {\tau_{zx}}^2 + {\tau_{xy}}^2\right)}$$

And the general effective strain

$$\varepsilon_{eff} = \frac{\sqrt{2}}{2(1+\nu)}\sqrt{\left(\varepsilon_x - \varepsilon_y\right)^2 + \left(\varepsilon_y - \varepsilon_z\right)^2 + (\varepsilon_z - \varepsilon_x)^2 + \frac{3}{2}\left({\gamma_{yz}}^2 + {\gamma_{zx}}^2 + {\gamma_{xy}}^2\right)}$$

And

$$E_{sec} = \frac{\sigma_{eff}}{\varepsilon_{eff}}$$

The equalization of the σ_{eff}- ε_{eff} curve to the uniaxial $\sigma - \varepsilon$ curve when the stresses applied to the material are reduced to uniaxial state of stress enables us to determine the E_{sec} empirically and solve the deformation theory problem:

$$\sigma_{eff} = \frac{\sqrt{2}}{2}\sqrt{(\sigma)^2 + (\sigma)^2} = \sigma$$

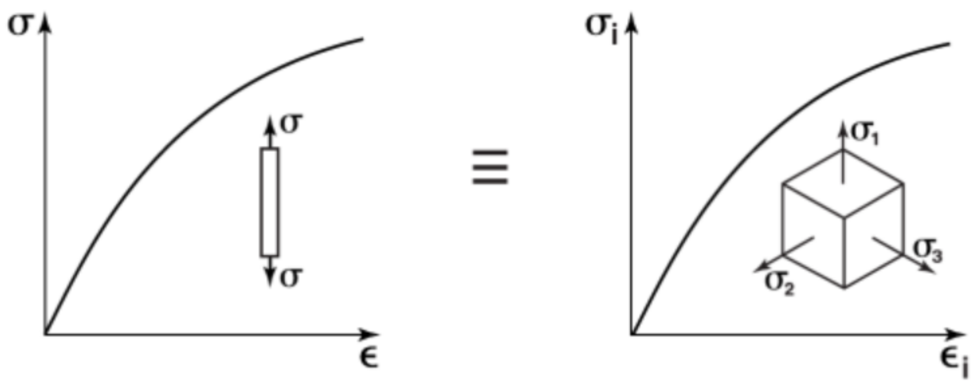
And

$$\varepsilon_{eff} = \frac{\sqrt{2}}{2(1+\nu)}\sqrt{\varepsilon^2(1+\nu)^2 + [-\nu\varepsilon - (-\nu\varepsilon)]^2 + \varepsilon^2(-\nu - 1)^2} = \varepsilon$$

Hence

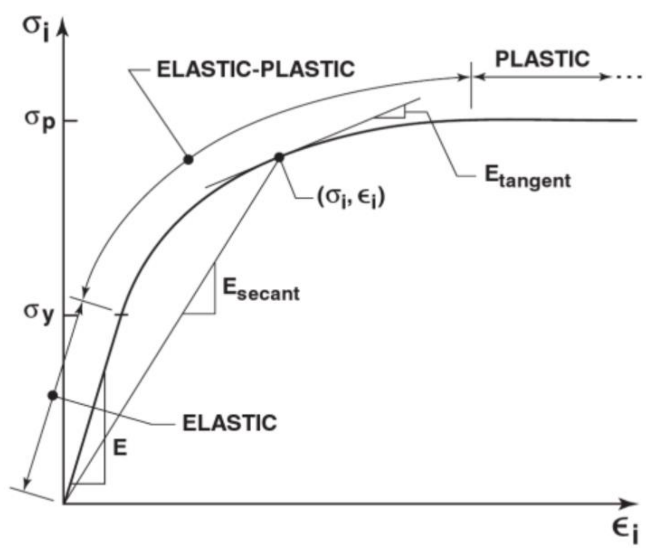
$$E_{sec} = \frac{\sigma}{\varepsilon}$$

Relating the material properties E_{sec}, ν for deformation under a multiaxial stress state to the properties determined in the usual uniaxial mechanical characterization test is important especially for materials with nonlinear stress-strain behavior as their properties are a nonlinear function of all the multiaxial stresses that act.



a Uniaxial $\sigma - \varepsilon$ Curve   b Universal $\sigma_i - \varepsilon_i$ Curve

The $\sigma - \varepsilon$ curve obtained from the uniaxial test is the same as the σ_{eff}- ε_{eff} curve which represents all multiaxial stress. This remarkable identity makes the uniaxial stress-strain curve the universal stress-strain curve



To obtain the secant modulus form a uniaxial stress-strain curve, the curve should be considered as a set of points each of which is a stress and corresponding strain that are obtained during a measurement.

- Enter the stress-strain curve at a specific value of σ_{eff} corresponding to the given multiaxial stress state
- Determine E_{sec} either graphically by drawing a straight line to the origin or by interpolation of the stress values from a table of stress-strain data pairs.
- Use E_{sec} and the variable ν in the stress-strain relations to calculate the strains for the specified σ_{eff}
- The strains $\varepsilon_x, \varepsilon_y, \varepsilon_z,$ etc. are the answers we are looking for, not ε_{eff} that can be obtained directly from the curve. Use the strain-stress equations to calculate them:

$$\varepsilon_x = \frac{1}{E_{sec}}\left(\sigma_x - \nu\left(\sigma_y + \sigma_z\right)\right)$$

$$\varepsilon_y = \frac{1}{E_{sec}}\left(\sigma_y - \nu(\sigma_x + \sigma_z)\right)$$

$$\varepsilon_z = \frac{1}{E_{sec}}\left(\sigma_z - \nu\left(\sigma_x + \sigma_y\right)\right)$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G^p}$$

$$\gamma_{yz} = \frac{\tau_{yz}}{G^p}$$

$$\gamma_{zx} = \frac{\tau_{zx}}{G^p}$$

There are five model equations to represent the stress-strain curves of common strain-hardening materials:

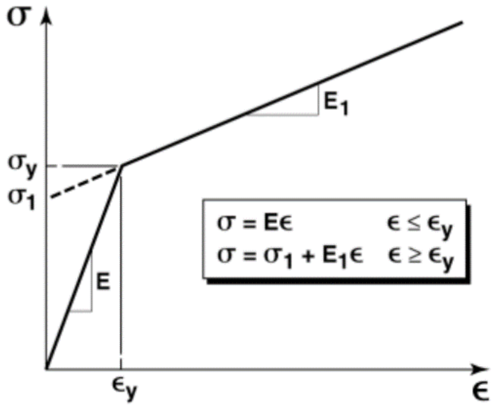
1. Linear strain hardening
2. Power-law
3. Ramberg-Osgood
4. Nadai
5. Nadai-Jones

These models cover more than one class of strain-hardening materials in addition to the non-strain hardening elastic-perfectly plastic and rigid-perfectly plastic curves.

The least number of parameters to describe elastic-perfectly plastic stress-strain behavior is 2: E and σ_y

At least three parameters will be needed to approximate non-linear stress-strain behavior using these models.

Linear strain-hardening model is the simplest of all non-linear stress-strain curve models. It consists of two straight lines with different moduli:



$$\sigma = E\varepsilon \qquad \varepsilon \le \varepsilon_y$$
$$\sigma = \sigma_1 + E_1\varepsilon \qquad \varepsilon > \varepsilon_y$$

It becomes the elastic-perfectly plastic model when $E_1 = 0$ and $\sigma_1 = \sigma_y$,
and the linear elastic model when $E_1 = E$ and $\sigma_1 = 0$

Three parameters are necessary to determine the strains as functions of stress: $E \ , E_1, \sigma_y$

The elastic and the first moduli are determined graphically from the secant modulus:

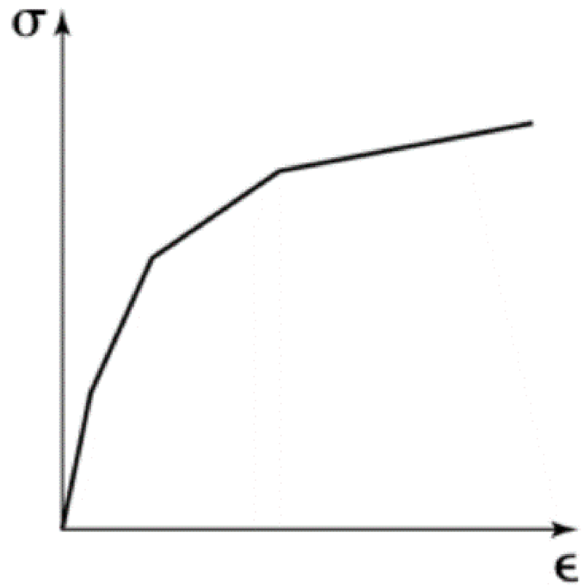
$$E_{sec} = E \qquad \sigma \le \sigma_y$$
$$E_{sec} = \frac{\sigma_1}{\varepsilon} + E_1 \qquad \sigma \ge \sigma_y$$

The stress at yielding is written by the use of the equation of a straight line y=mx+b:

$$\sigma_y = \sigma_1 + E_1\varepsilon_y$$

The stress-strain behavior of many materials can be represented roughly with this model.

Philip’s model is a mathematical extension of the linear strain-hardening model which consists of multiple straight line segments.



The more line segments that exist, the better the measured stress-strain behavior can be modeled.
However the mathematical difficulty increases with the addition of each line segment as more parameters are introduced in the model equation.

Power-law model

The general form of power-law stress-strain curve model has the following equation

$$\sigma = A\varepsilon^n$$

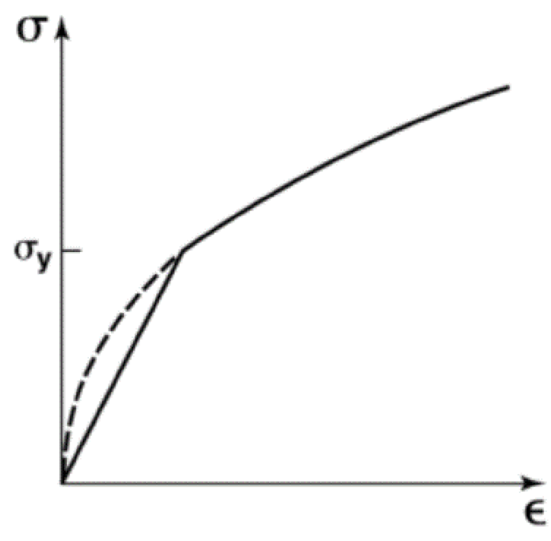
N is the strain hardening coefficient, A is the constant which are adjusted to best fit measured stress-strain data.

The value of n should be in the range 0-1 in order to model concave-downward behavior.

The stress-strain curve has an infinite slop at the origin and the equation is not good for low stress levels. Instead the following form is used to account for elastic deformations:

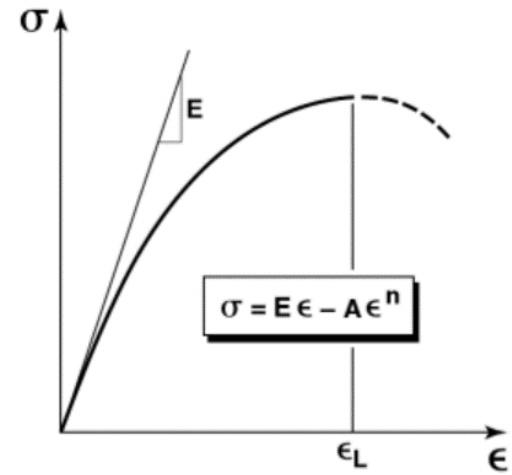
$$\sigma = E\varepsilon \qquad \sigma \leq \sigma_y$$

$$\sigma = A\varepsilon^n \qquad \sigma \geq \sigma_y$$



An alternative form is

$$\sigma = E\varepsilon - A\varepsilon^n$$



Only three stress-strain curve parameters are needed for this equation: E, A, n

It is valid until the maximum stress-strain point corresponding to ε_L

$$\varepsilon_L = \left(\frac{E}{An}\right)^{\frac{1}{n-1}}$$

Power-law model is used extensively because of its mathematical simplicity, however only certain types of stress-strain behavior can be modeled with it.

Ramberg-Osgood Model

The general form of Ramberg-Osgood stress-strain curve equation is

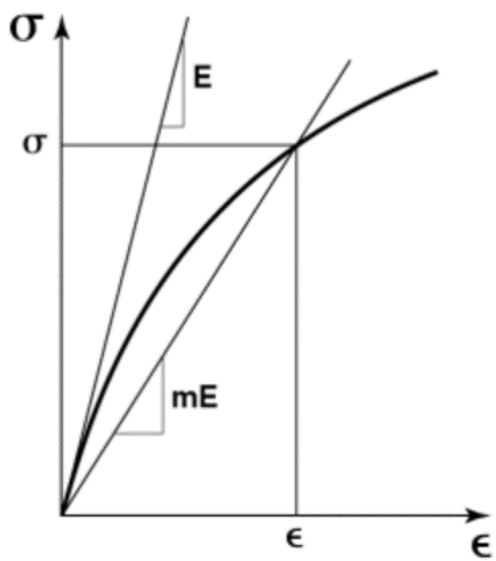
$$\varepsilon = \frac{\sigma}{E} + K\left(\frac{\sigma}{E}\right)^{n}$$

The first part in the right hand side is the elastic strain and the second is the plastic strain

K is a constant depending on the modulus E, curve fitting parameter n and m which is a proportionality constant:

$$K = \left(\frac{1-m}{m}\right)\left(\frac{E}{\sigma_m}\right)^{n-1}$$

Where m is the proportion of the offset yield stress of aluminum to the stress in material under consideration:

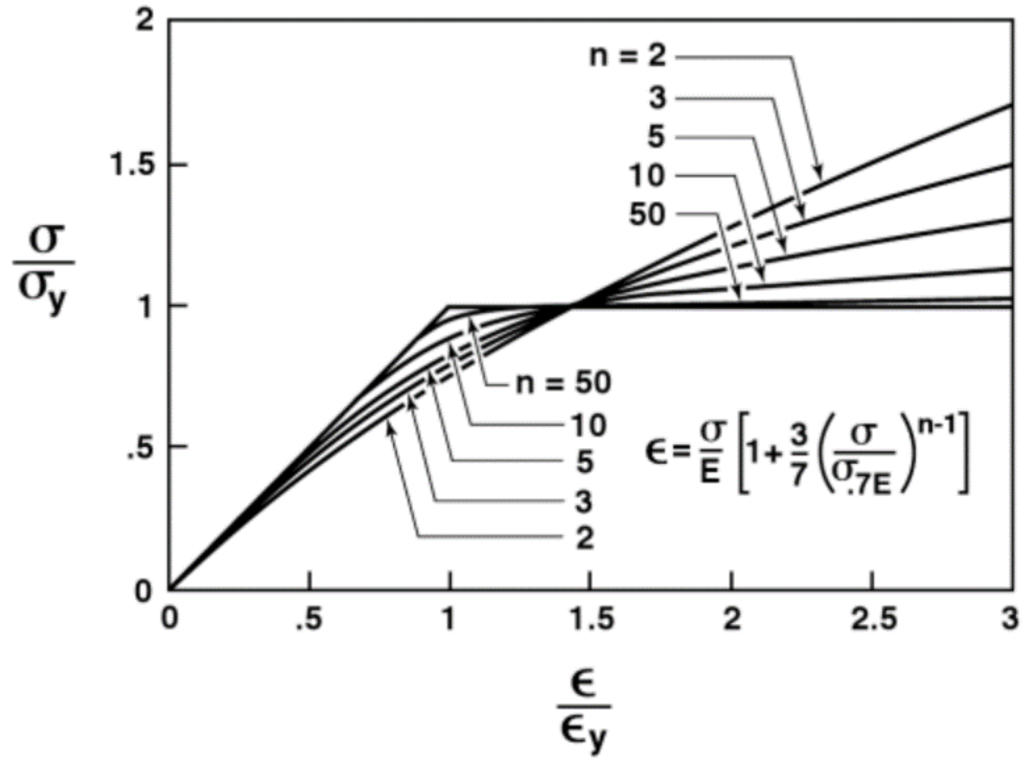


$$\sigma = mE\varepsilon$$

It is around 0.7 for aluminum

$$\varepsilon = \frac{\sigma}{E}\left(1 + \frac{0.3}{0.7}\left(\frac{\sigma}{\sigma_{0.7}}\right)^{n-1}\right)$$

Three parameters are needed to determine the Ramberg-Osgood stress-strain curve: E, $\sigma_{0.7E}$ and n



The model equation is continuously curved, there is no definitive elastic region followed by a yield stress. It approaches elastic-perfectly plastic behavior as n gets larger

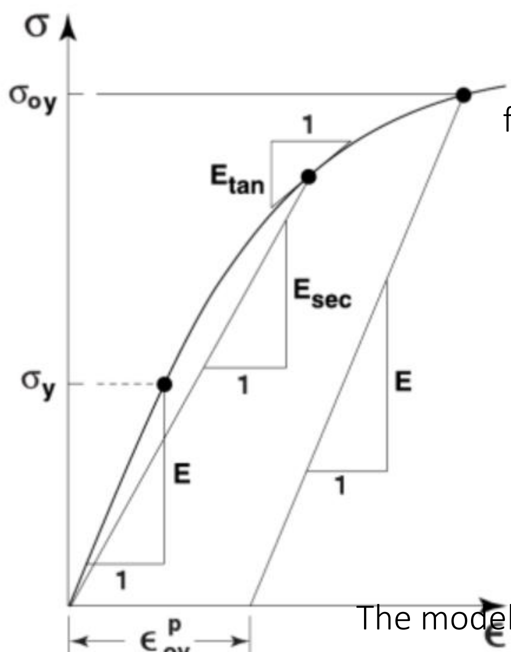
Nadai Model

The behavior of elastic-plastic materials like aluminum and its alloys are represented well with a linear elastic region ended by a well defined toeld stress and a gradual bending over of the conceave downward stress-strain curve.
Nadai model equation represents this behavior:

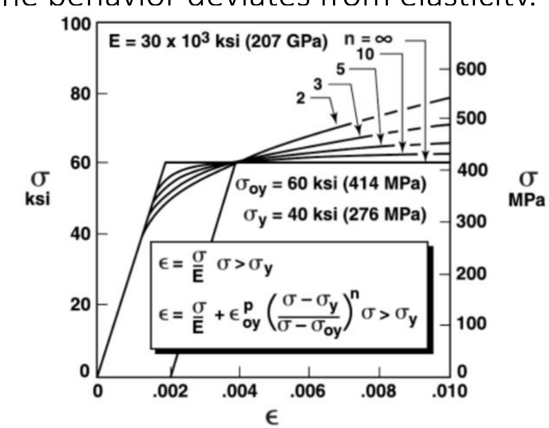
$$\varepsilon = \frac{\sigma}{E} \qquad \sigma \le \sigma_y$$
$$\varepsilon = \frac{\sigma}{E} + K(\sigma - \sigma_y)^n \qquad \sigma \ge \sigma_y$$

Where K is constant dependent on the fitting parameter n, the off-set yield strain $\varepsilon_{oy} = 0.002,$ and the stress at the off-set yield point:

$$K = \varepsilon_{oy}{}^p(\sigma - \sigma_{oy})^{-n}$$



The off-set yield strain at 0.002 is determined from the permanent strain for materials like steel and aluminum where the behavior deviates from elasticitv.



Nadai maodel needs four parameters above the yield stress: E, σ_y, σ_{oy}, n
The model equation is similar to the Ramberg-Osgood equation. Both are same when σ_y=0

Nadai-Jones Model

The concept of Nadai stress-strain curve model is extended to cover plastic materials with two distinctly different regions of nonlinear behavior. Nadai-Jones equation is the same until an upper stress where a second highly nonlinear region is reached:

$$\varepsilon = \frac{\sigma}{E} \qquad \sigma \le \sigma_y$$

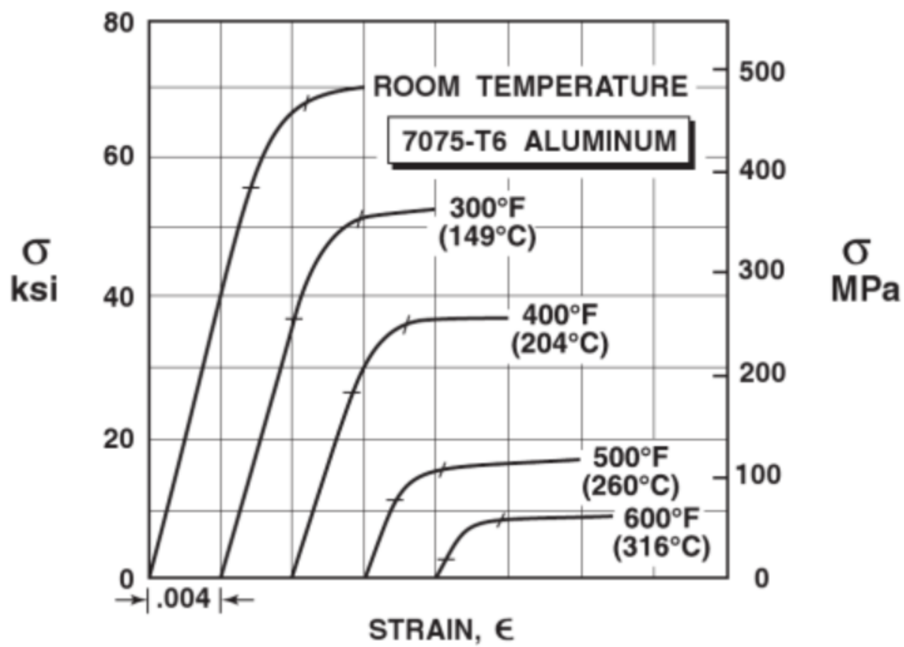
$$\varepsilon = \frac{\sigma}{E} + K\left(\sigma - \sigma_y\right)^n \qquad \sigma_2 \ge \sigma \ge \sigma_y$$

$$\varepsilon = \frac{\sigma}{E} + K\left(\sigma - \sigma_y\right)^n + J(\sigma - \sigma_2)^m \qquad \sigma_2 \ge \sigma \ge \sigma_y$$

Where K and J are constants that depend on upper stresses σ_1 and σ_3 ,the corresponding plastic strains ${\varepsilon_1}^p$ and ${\varepsilon_3}^p$and the curve fitting constants n and m

$$K = {\varepsilon_1}^p\left(\sigma_1 - \sigma_y\right)^{-n}$$
$$J = {\varepsilon_3}^p\left(\sigma_3 - \sigma_y\right)^{-n}$$

Analyze the uniaxial deformation of aluminum alloy using different models and determine the strains if the normal stresses in the x and y direction are 70 MPa and 40 MPa, and the shearing stress on the plane is 30 MPa



Compressive Stress-Strain Curves for 7075-T6 Aluminum as a Function of Temperature *(After Mathauser and Brooks*