Plasticity and Deformation Process

Stress-strain relations in plasticity

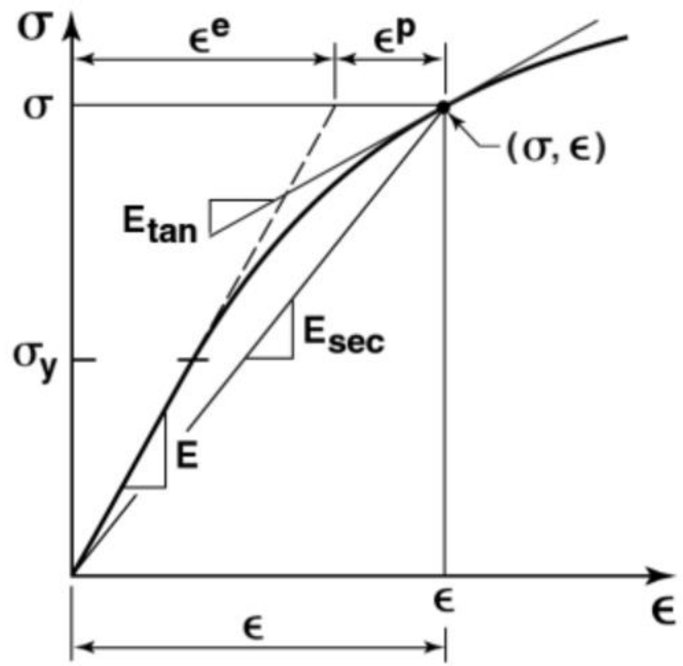
and the Deformation theory

The fundamental problem in the solution of a plasticity problem is to determine how stresses and strains can be found for a specified state of loading on a body.

There are two theories to describe the relation between stresses and strains:

Deformation or total strain theory:
Total strains are directly related to the total stresses by the secant modulus which is a function of the stress level
The strains on an object depend on the final state of stress, they are independent of stress history

Flow or incremental strain theory:
Increments of plastic strain $\Delta\varepsilon^P$ are related to increments of plastic stress $\Delta\sigma^P$ by the tangent modulus which is a function of the stress level



The basis for the deformation theory of plasticity is the elastic stress-strain relations and the associated stress and strain intensities for multiaxial stress states.

The stress intensity or the effective stress for an elastic material is expressed as

$$\sigma_{eff} = \frac{\sqrt{2}}{2}\sqrt{(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6({\tau_{yz}}^2 + {\tau_{zx}}^2 + {\tau_{xy}}^2)}$$

And the effective strain as

$$\varepsilon_{eff} = \frac{\sqrt{2}}{2(1+\nu)}\sqrt{(\varepsilon_x - \varepsilon_y)^2 + (\varepsilon_y - \varepsilon_z)^2 + (\varepsilon_z - \varepsilon_x)^2 + \frac{3}{2}({\gamma_{yz}}^2 + {\gamma_{zx}}^2 + {\gamma_{xy}}^2)}$$

And

$$\sigma_{eff} = E\varepsilon_{eff}$$

The three dimensional elastic stress-strain relations for an isotropic material in terms of Young’s modulus, and Poisson’s ratio are:

$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_x + \nu(\varepsilon_y + \varepsilon_z)\right]$$

$$\sigma_y = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_y + \nu(\varepsilon_z + \varepsilon_x)\right]$$

$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_z + \nu(\varepsilon_x + \varepsilon_y)\right]$$

$$\tau_{yz} = \frac{E}{2(1+\nu)}\gamma_{yz}$$

$$\tau_{zx} = \frac{E}{2(1+\nu)}\gamma_{zx}$$

$$\tau_{xy} = \frac{E}{2(1+\nu)}\gamma_{xy}$$

Most deformation processes involving thin plates of material are approximated to the plane stress conditions

Plane stress is a state of stress in which the normal stress σ_z, and the shear stresses σ_{xz}, σ_{yz} directed perpendicular to the x-y plane are assumed to be zero

The geometry of the body is that of a plate with one dimension much smaller than the others. The loads are applied uniformly over the thickness of the plate and act in the plane of the plate as shown.

$$\{\sigma\} = [D]\,\{\varepsilon\}$$

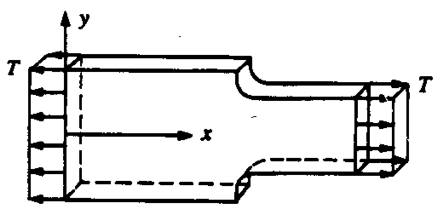


plate with fillet

$$[D] = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

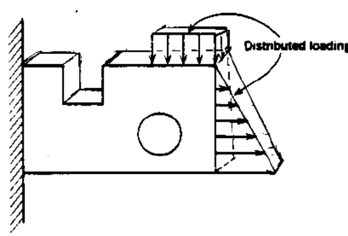


plate with hole

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix}\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix}$$

The plane stress condition is the simplest form of behavior for continuum structures and represents situations frequently encountered in practice

Under plane stress conditions $\sigma_z = 0$, $\tau_{xz} = \tau_{yz} = 0$ and $\varepsilon_z = -\frac{\nu}{1-\nu}(\varepsilon_x + \varepsilon_y)$

So the effective strain for a state of plane stress is

$$\varepsilon_{eff} = \frac{\sqrt{2}}{2(1-\nu^2)}\sqrt{(1-\nu+\nu^2)(\varepsilon_x{}^2+\varepsilon_y{}^2)+(1-4\nu+\nu^2)\varepsilon_x\varepsilon_y+\frac{3}{4}(1-\nu)^2\gamma_{xy}{}^2}$$

And the effective stress is

$$\sigma_{eff} = \frac{\sqrt{2}}{2}\sqrt{(\sigma_x-\sigma_y)^2+(\sigma_y)^2+(\sigma_z)^2+6(\tau_{xy}{}^2)}$$

Example – A thin disk of aluminum is stressed under plane stress conditions. Determine the effective stress and strain resulting from the load if the normal stresses in the x and y direction are 70 MPa and 40 MPa, and the shearing stress on the plane is 30 MPa. E_{Al}= 70 GPa

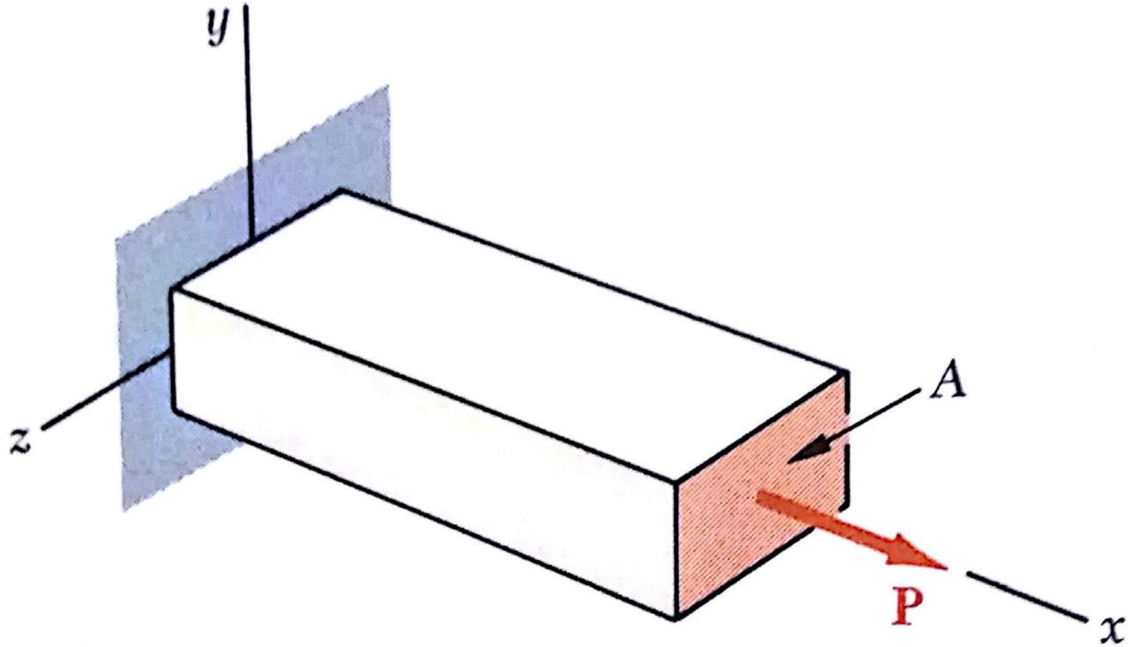
In total strain or incremental strain theories, the strains are separated into an elastic component ε^e and a plastic component ε^P

The elastic component of strain under uniaxial stress loading is σ/E

The plastic component is represented by a complex equation as will be derived next

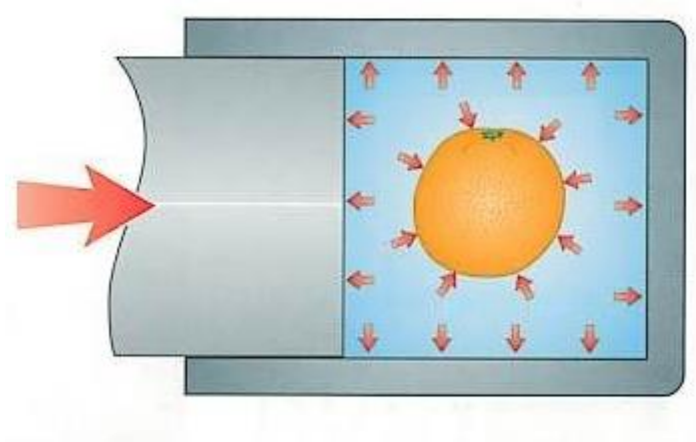
Both deformation and incremental theories are based on the assumption that elastic deformation is compressible and plastic deformation is incompressible

The compressible nature of elastic deformation is obvious from the fact that Poisson's ratio for ordinary isotropic materials is much less than one-half (0-0.35)



The incompressibilty of plastic deformation is not obvious

Experiments on materials subjected to very high hydrostatic pressures show that the density and volume do not significantly change under extremely high pressures and the change is elastic not plastic



The dilatation or volumetric strain of materials under hydrostatic stress is

$$\theta = \varepsilon_x + \varepsilon_y + \varepsilon_z = \varepsilon(1 - 2\nu)$$

In which ε is the total strain under uniaxial stress σ and ν is the Poisson's ratio for total strains

Dilatation can be regarded as the sum of an elastic dilatation and a plastic dilatation

$$\theta^e = \varepsilon^e(1 - 2\nu^e) \qquad \theta^p = \varepsilon^p(1 - 2\nu^p)$$

$$\theta = \theta^e + \theta^p = \varepsilon(1-2\nu) = \varepsilon^e(1-2\nu^e) + \varepsilon^p(1-2\nu^p)$$

Dividing both sides by ε and substituting $\varepsilon - \varepsilon^e$ for ε^p gives:

$$(1-2\nu) = \frac{\varepsilon^e}{\varepsilon}(1-2\nu^e) + \frac{\varepsilon - \varepsilon^e}{\varepsilon}(1-2\nu^p)$$

$$-2\nu = \frac{\varepsilon^e}{\varepsilon}(-2\nu^e) - 2\nu^p + \frac{\varepsilon^e}{\varepsilon}2\nu^p$$

So

$$\nu = \nu^p - \frac{\varepsilon^e}{\varepsilon}(\nu^p - \nu^e)$$

The strains can be expressed in terms of the moduli:

$$\frac{\varepsilon^e}{\varepsilon} = \frac{\sigma/E}{\sigma/E_{sec}} = \frac{E_{sec}}{E}$$

Finally

$$\nu = \nu^p - \frac{E_{sec}}{E}(\nu^p - \nu^e)$$

Or

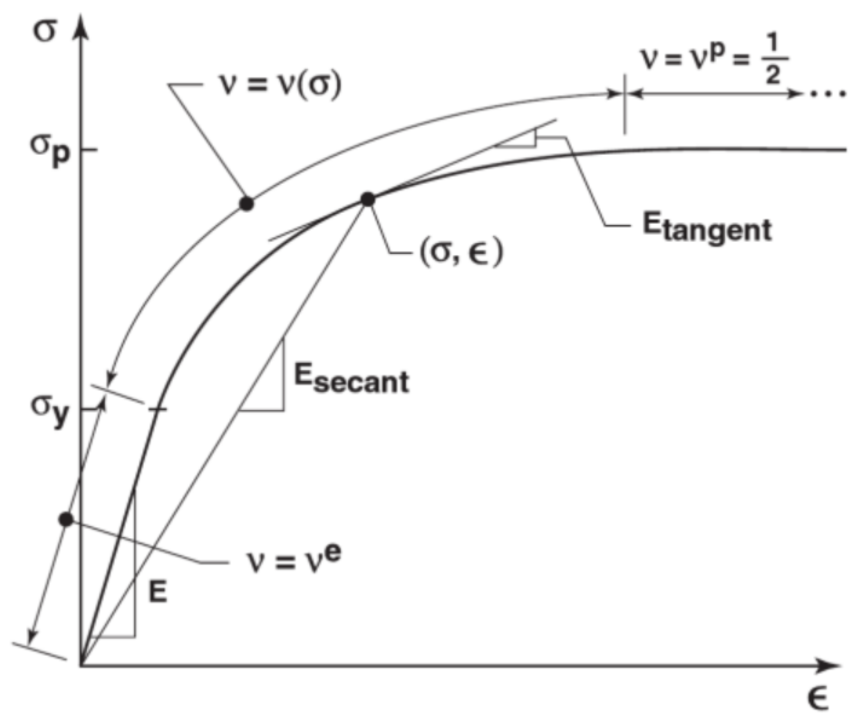
$$\nu = \frac{1}{2} - \frac{E_{sec}}{E}\left(\frac{1}{2} - \nu^e\right)$$

Since materials are incompressible during plastic deformation, the Poisson's ratio changes from the elastic value to the incompressible value ½ in a very gradual way as the stress is increased above the yield stress

$$\nu = \nu^e \qquad \text{for } \sigma \leq \sigma_y$$

$$\nu = \nu(\sigma) = \frac{1}{2} - \frac{E_{sec}}{E}\left(\frac{1}{2} - \nu^e\right) \qquad \text{for } \sigma > \sigma_y$$

$$\nu = \frac{1}{2} \qquad \text{for } \sigma \gg \sigma_y$$



Stress-strain relations according to the deformation theory

Strains are separated by Hencky et al. into elastic and plastic strains:

$$\varepsilon_x = {\varepsilon_x}^e + {\varepsilon_x}^p$$

$$\gamma_{xy} = {\gamma_{xy}}^e + {\gamma_{xy}}^p$$

The elastic strains are obtained from the Hooke's law

$${\varepsilon_x}^e = \frac{1}{E}\left(\sigma_x - \nu^e\left(\sigma_y + \sigma_z\right)\right)$$

$${\varepsilon_y}^e = \frac{1}{E}\left(\sigma_y - \nu^e(\sigma_x + \sigma_z)\right)$$

$${\varepsilon_z}^e = \frac{1}{E}\left(\sigma_z - \nu^e\left(\sigma_x + \sigma_y\right)\right)$$

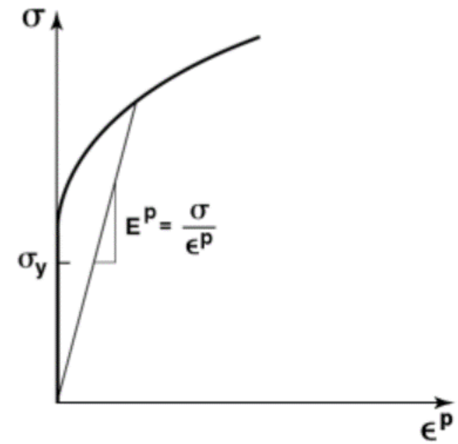
$${\gamma_{xy}}^e = \frac{\tau_{xy}}{G}$$

$${\gamma_{yz}}^e = \frac{\tau_{yz}}{G}$$

$${\gamma_{zx}}^e = \frac{\tau_{zx}}{G}$$

Where the mechanical properties E and G are the normal elastic modulus and shear modulus

To obtain the plastic moduli we need to consider the stress-strain diagram in terms of the normal stress and the **plastic strain**



The plastic modulus at any stress above the yield stress is the secant modulus at that point

So

$$\varepsilon_x{}^p = \frac{1}{E^p}\left(\sigma_x - \nu^p\left(\sigma_y + \sigma_z\right)\right)$$
$$\varepsilon_y{}^p = \frac{1}{E^p}\left(\sigma_y - \nu^p(\sigma_x + \sigma_z)\right)$$
$$\varepsilon_z{}^p = \frac{1}{E^p}\left(\sigma_z - \nu^p\left(\sigma_x + \sigma_y\right)\right)$$
$$\gamma_{xy}{}^p = \frac{\tau_{xy}}{G^p}$$
$$\gamma_{yz}{}^p = \frac{\tau_{yz}}{G^p}$$
$$\gamma_{zx}{}^p = \frac{\tau_{zx}}{G^p}$$

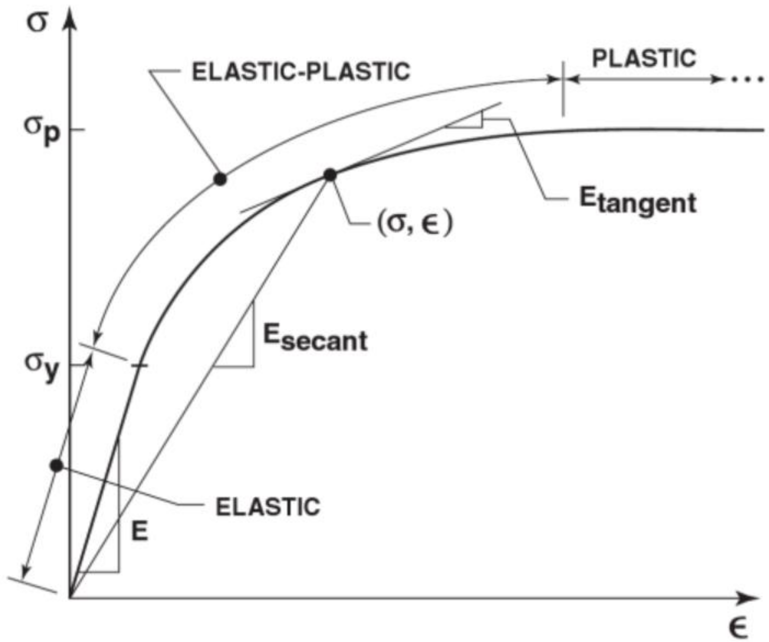
Where ν^p is the plastic Poisson's ratio (1/2) and G^p is the plastic shear modulus:

$$G^p = \frac{E^p}{2(1+\nu)} = \frac{E^p}{3}$$

The elastic and plastic strains sum to the total strains

$$\frac{1}{E_{sec}} = \frac{1}{E} + \frac{1}{E^p}$$

$$\nu = \frac{\nu^e/E + \nu^p/E^p}{1/E + 1/E^p}$$



The stress-strain behavior is divided into three regions: elastic, elastic-plastic and plastic

Only the elastic-plastic region is considered in the deformation theory as a material nonlinearity because plastic poisson's ratio is constant

Partitioning of the strains into elastic and plastic components is used in derivation of the deformation theory. Deformation theory enables us determine the strains from a state of multiaxial stress, in a single form that applies to all three deformation regions

These total stress-strain relations for an isotropic material in terms of the secant modulus and a continuously variable Poisson's ratio are similarly:

$$\sigma_x = \frac{E_{secant}}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_x + \nu\left(\varepsilon_y + \varepsilon_z\right)\right]$$

$$\sigma_y = \frac{E_{secant}}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_y + \nu(\varepsilon_z + \varepsilon_x)\right]$$

$$\sigma_x = \frac{E_{secant}}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_z + \nu\left(\varepsilon_x + \varepsilon_y\right)\right]$$

$$\tau_{yz} = \frac{E_{secant}}{2(1+\nu)}\gamma_{yz} = G_{secant}\gamma_{yz}$$

$$\tau_{zx} = \frac{E_{secant}}{2(1+\nu)}\gamma_{zx} = G_{secant}\gamma_{zx}$$

$$\tau_{xy} = \frac{E_{secant}}{2(1+\nu)}\gamma_{xy} = G_{secant}\gamma_{xy}$$

Where

$$\nu = \frac{1}{2} - \frac{E_{sec}}{E}\left(\frac{1}{2} - \nu^e\right)$$

And

$$G_{secant} = \frac{E_{sec}}{2(1+\nu)}$$

Deformation theory is workable only if E_{sec} and ν are expressed as a function of the multiaxial stress state

$$\nu = \frac{1}{2} - \frac{E_{sec}}{E}\left(\frac{1}{2} - \nu^e\right)$$

The E_{sec} should be expressed as a function of the multiaxial stress state

The general effective stress equation is

$$\sigma_{eff} = \frac{\sqrt{2}}{2}\sqrt{\left(\sigma_x - \sigma_y\right)^2 + \left(\sigma_y - \sigma_z\right)^2 + (\sigma_z - \sigma_x)^2 + 6\left(\tau_{yz}{}^2 + \tau_{zx}{}^2 + \tau_{xy}{}^2\right)}$$

And the general effective strain

$$\varepsilon_{eff} = \frac{\sqrt{2}}{2(1+\nu)}\sqrt{\left(\varepsilon_x - \varepsilon_y\right)^2 + \left(\varepsilon_y - \varepsilon_z\right)^2 + (\varepsilon_z - \varepsilon_x)^2 + \frac{3}{2}\left(\gamma_{yz}{}^2 + \gamma_{zx}{}^2 + \gamma_{xy}{}^2\right)}$$

And

$$E_{sec} = \frac{\sigma_{eff}}{\varepsilon_{eff}}$$

The equalization of the σ_{eff}- ε_{eff} curve to the uniaxial $\sigma - \varepsilon$ curve (the stresses applied to the material are reduced to uniaxial state of stress) enables us to determine the E_{sec} empirically and solve the deformation theory problem:

$$\sigma_{eff} = \frac{\sqrt{2}}{2}\sqrt{(\sigma)^2 + (\sigma)^2} = \sigma$$

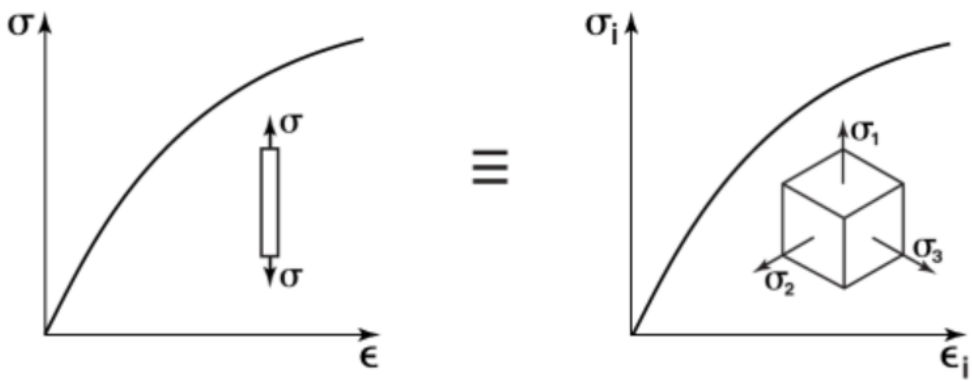
And

$$\varepsilon_{eff} = \frac{\sqrt{2}}{2(1+\nu)}\sqrt{\varepsilon^2(1+\nu)^2 + [-\nu\varepsilon - (-\nu\varepsilon)]^2 + \varepsilon^2(-\nu-1)^2} = \varepsilon$$

Hence

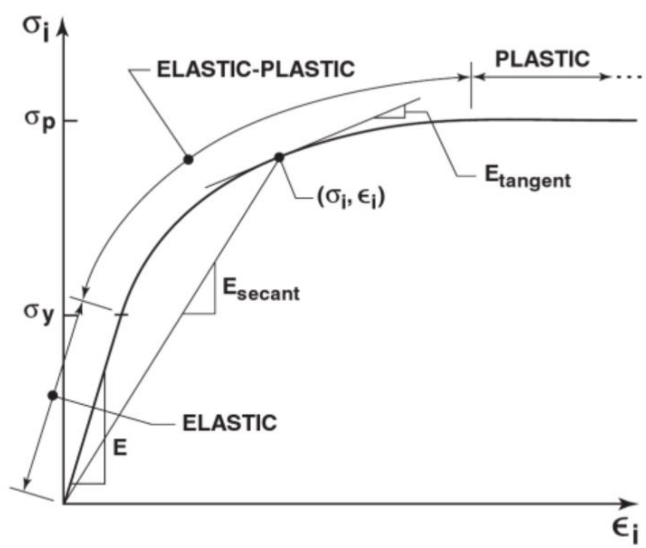
$$E_{sec} = \frac{\sigma}{\varepsilon}$$

Relating the material properties E_{sec}, ν for deformation under a multiaxial stress state to the properties determined in the usual uniaxial mechanical characterization test is important especially for materials with nonlinear stress-strain behavior as their properties are a nonlinear function of all the multiaxial stresses that act.



a Uniaxial $\sigma - \varepsilon$ Curve b Universal $\sigma_i - \varepsilon_i$ Curve

The $\sigma - \varepsilon$ curve obtained from the uniaxial test is the same as the σ_{eff}- ε_{eff} curve which represents all multiaxial stresses. This remarkable identity makes the uniaxial stress-strain curve the universal stress-strain curve



Procedure to obtain the secant modulus form a uniaxial stress-strain curve:

Basically the curve should be considered as a set of points each of which is a stress and corresponding strain that are obtained during a measurement.

- Mark the stress-strain curve at a specific value of σ_{eff} corresponding to the given multiaxial stress state
- Determine E_{sec} graphically by drawing a straight line to the origin.
- Use E_{sec} and the variable ν in the stress-strain relations to calculate the strains for the specified σ_{eff}
- The strains $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$ that are given below are the answers we are looking for, not ε_{eff} that can be obtained directly from the curve.

$$\varepsilon_x = \frac{1}{E_{sec}}\left(\sigma_x - \nu\left(\sigma_y + \sigma_z\right)\right)$$

$$\varepsilon_y = \frac{1}{E_{sec}}\left(\sigma_y - \nu(\sigma_x + \sigma_z)\right)$$

$$\varepsilon_z = \frac{1}{E_{sec}}\left(\sigma_z - \nu\left(\sigma_x + \sigma_y\right)\right)$$

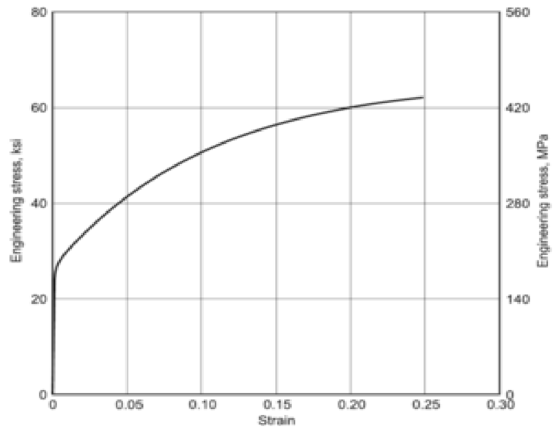
$$\gamma_{xy} = \frac{\tau_{xy}}{G^p}$$

$$\gamma_{yz} = \frac{\tau_{yz}}{G^p}$$

$$\gamma_{zx} = \frac{\tau_{zx}}{G^p}$$

Example – Use the stress-strain diagram for Nickel ($\nu = 0.31$) to calculate its elastic modulus and secant modulus for an effective stress state of 400 MPa.

Nickel (Ni)



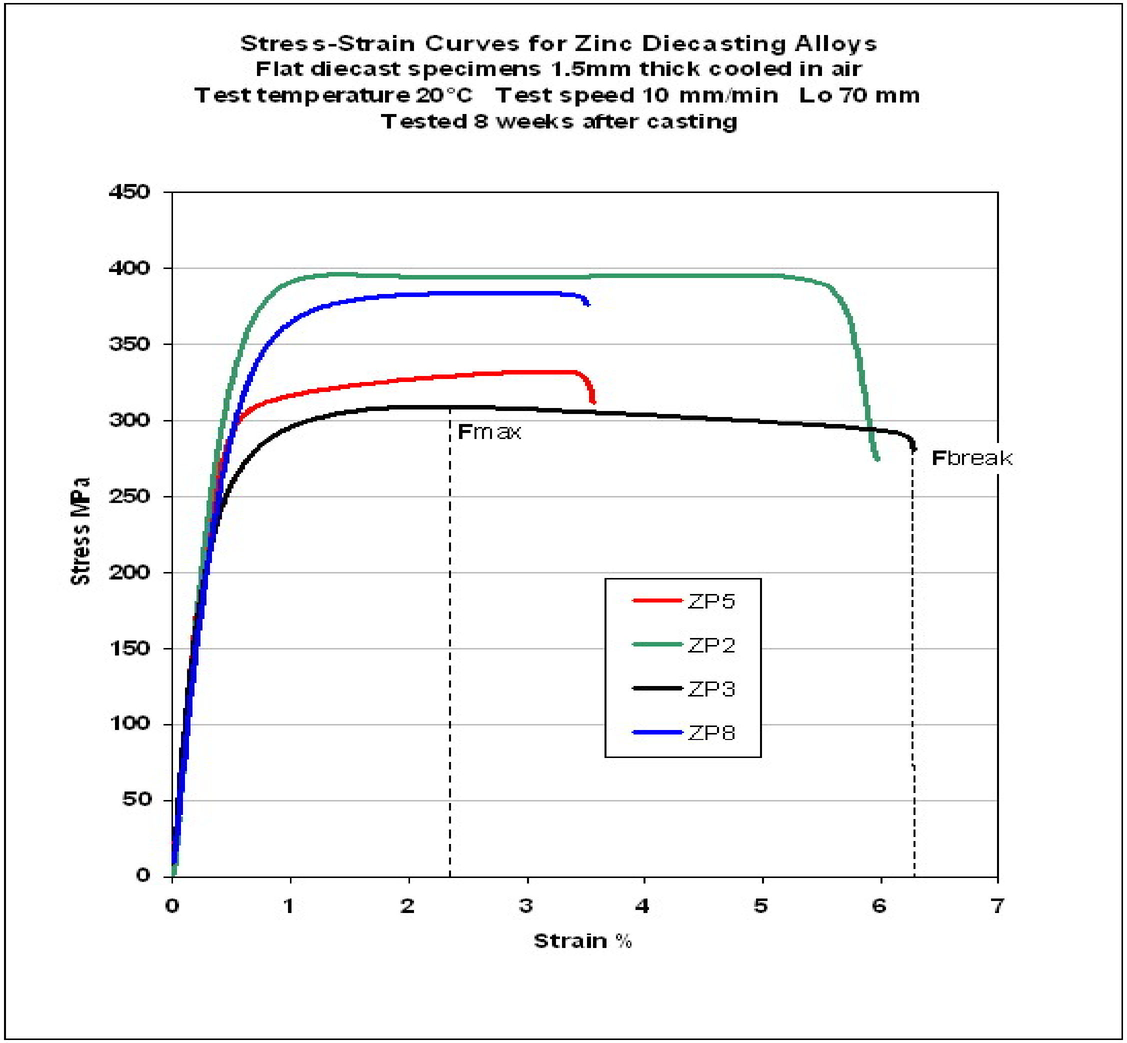
Ni.001 Ni 200 annealed nickel sheet, engineering stress-strain curve (full range)

Test direction: longitudinal. Sheet thickness = 0.787 mm (0.031 in.). Commercially pure nickel (UNS N02200). 0.2% yield strength = 185 MPa (26.9 ksi); ultimate tensile strength = 434 MPa (63.0 ksi); elongation = 39.5%; strength coefficient (K) = 138.2; strain-hardening exponent (n) = 0.387. Composition: Ni 99.0 min

Courtesy of Special Metals Corporation

Example – Use the stress-strain diagram for Zinc ($\nu = 0.29$) to calculate its elastic modulus, secant modulus and resulting strains for a plane stress state of

$$\sigma_x = 70\ MPa, \sigma_x = 50\ MPa, \tau_{xy} = 60\ MPa$$



The modulus is expressed as a function of the stresses using various material models.

Although the secant modulus is dynamic and depends on the stress level, the problem-solving becomes straight-forward with the help of stress-strain diagrams or models that include strain hardening

Hencky used the von Mises yield criterion and the distortional energy concept to derive the stress-strain relations

The combination of the yield criterion, the stress-strain relations, and the material model gives the complete deformation theory of plasticity

Deformation theory helps us predict the stresses and strains at a point on the stress-strain curve but does not consider the path taken to get there.

Because of that, loading and unloading can not be evaluated with the same material model using deformation theory and should be considered as separate events.

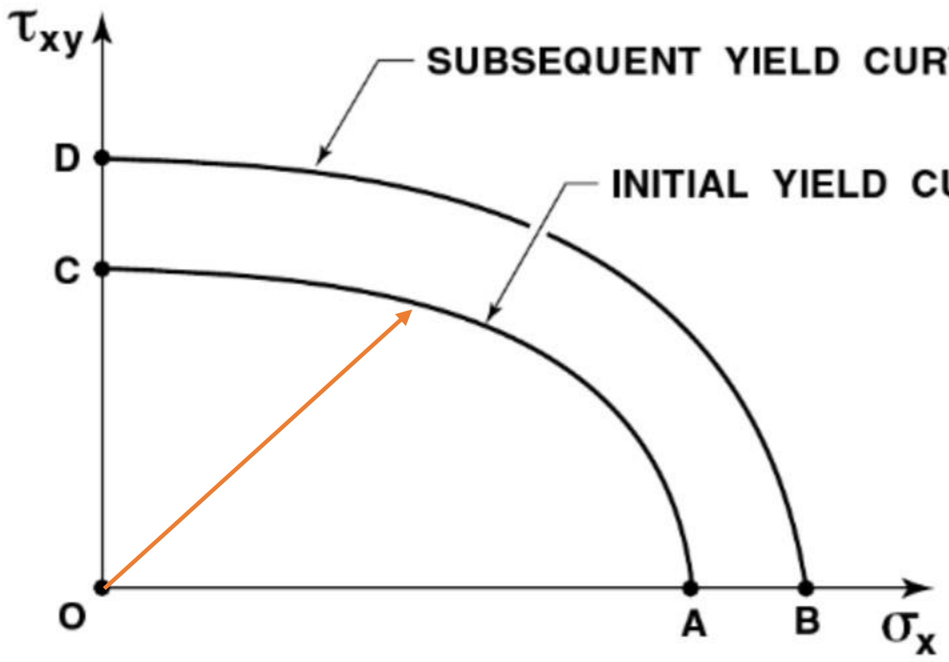
Another limitation of deformation theory is that all stresses in a multiaxial stress state must be applied in proportion to one another because deformation theory is not capable of distinguishing between types of loading.

These restrictions are valid for some plasticity problems and the theory is not generally applicable.

But it is applicable to most practical problems in metal forming and quite useful. Some problems that are easily solved with deformation theory are difficult to solve with incremental theory because of the excessively complex computation methods.

Proportional loading is when a material is loaded in such a way that the components of the deviatoric stress maintain proportionality throughout the load history

It is represented by a straight line passing through the origin in the principal stress space:



The components of deviatoric stresses for a proportional loading are represented as

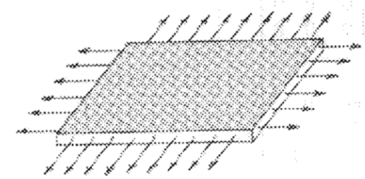
$$\sigma_{ij} = K * \sigma_{ij}^{0}$$

Where K is a monotonically increasing function (loading only) and σ_{ij}^{0} is an arbitrary state of stress

(a) Uniaxial stress : $\begin{pmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

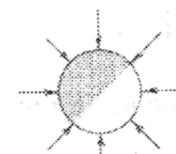
(a)

(b) Biaxial stress : $\begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$



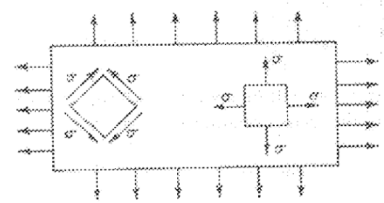
(b)

(c) Hydrostatic pressure : $\begin{pmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{pmatrix}$



(c)

(d) Pure shear : $\begin{pmatrix} 0 & \sigma & 0 \\ \sigma & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$



(d)

• Special Stress States •

Hydrostatic and Deviatoric Stresses

Total stress tensor can be divided into two components:

1. Hydrostatic or mean stress tensor (σ_m) involving only pure tension or compression
2. Deviatoric stress tensor (σ'_{ij}) representing pure shear with no normal components

$$\sigma_m = \frac{\sigma_{kk}}{3} = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3}$$

$$\sigma_{ij} = \sigma'_{ij} + \frac{1}{3}\delta_{ij}\sigma_{kk}$$

Example on Hydrostatic and Deviatoric Stresses

Given the stress state: $\sigma_{ij} = \begin{pmatrix} 80 & 20 & -50 \\ 20 & -40 & 30 \\ -50 & 30 & 50 \end{pmatrix}$,

a. *Find the hydrostatic part of the stresses.*

b. *Find the deviatoric part of the stresses.*

Ans. (a) $\sigma_{ij}^{hyd} = \sigma_m \delta_{ij} = \frac{1}{3}(\sigma_{11}+\sigma_{22}+\sigma_{33})\ \delta_{ij}$ where $\sigma_m = \frac{1}{3}(80 - 40 + 50) = 30$ so that

$$\sigma_{ij}^{hyd} = \begin{pmatrix} 30 & 0 & 0 \\ 0 & 30 & 0 \\ 0 & 0 & 30 \end{pmatrix}.$$

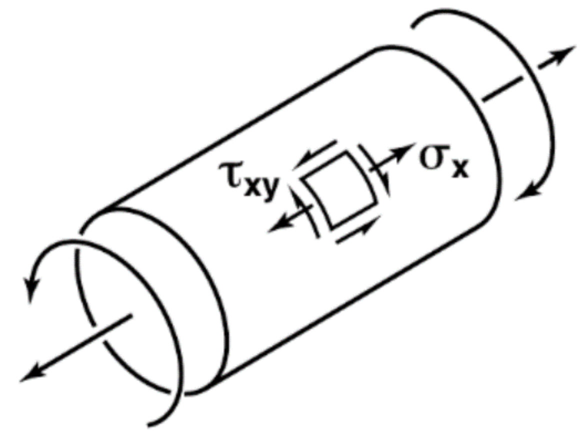
(b) By definition, $\sigma_{ij}^{dev} = \sigma_{ij} - \sigma_{ij}^{hyd} = \begin{pmatrix} 80 & 20 & -50 \\ 20 & -40 & 30 \\ -50 & 30 & 50 \end{pmatrix} - \begin{pmatrix} 30 & 0 & 0 \\ 0 & 30 & 0 \\ 0 & 0 & 30 \end{pmatrix} = \begin{pmatrix} 50 & 20 & -50 \\ 20 & -70 & 30 \\ -50 & 30 & 20 \end{pmatrix}$

Note that the mean hydrostatic stress for $\sigma_{ii}^{dev} = (\sigma_{11}^{dev} + \sigma_{22}^{dev} + \sigma_{33}^{dev}) = 0$, as expected.

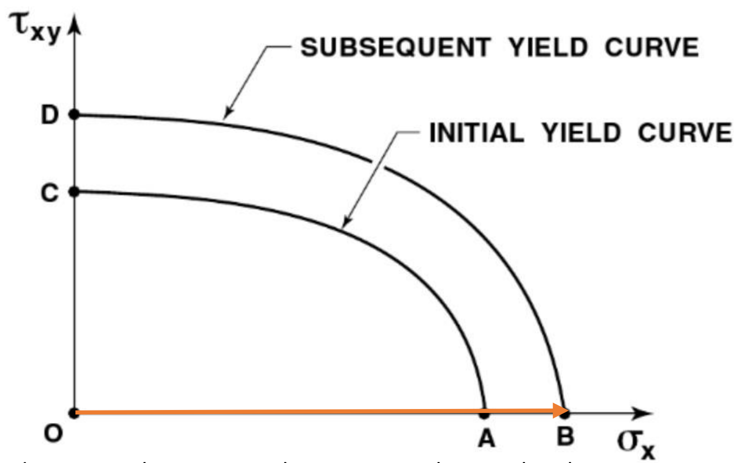
On the other hand, increments of strain are related to increments of stress as loading progresses in the **incremental strain theory**.
The stress history is accounted for by summing the stress increments (by integrating over the loading path to the final stress state)
The strains and stresses are related with a complicated nonlinear function that depends on the loading path

Stress history is important in deformation analysis because the final strain states for two identical final stress states may be different if they have different stress histories.

Consider a thin tube subjected to combinations of torsion and axial tension:



The loading in two dimensional stress space is represented for an isotropically hardening tube material as



Under simple uniaxial tension along the lone OAB, the material first yields at point A and then strain hardens to point B. The permanent plastic normal strains and the resulting elastic strains at point B are

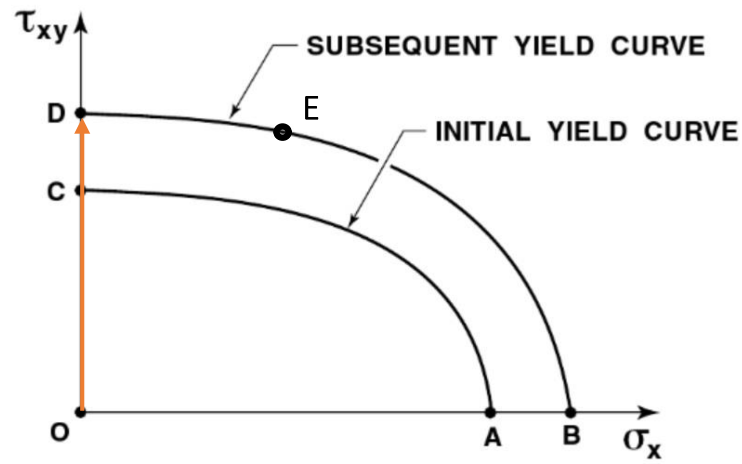
$$\varepsilon_x^p = \varepsilon^p \quad \varepsilon_y^p = \varepsilon_z^p = -\nu^p \varepsilon^p = -\frac{1}{2}\varepsilon^p \qquad \varepsilon_x^e = \varepsilon^e \quad \varepsilon_y^e = \varepsilon_z^e = -\nu^e \varepsilon^e$$

$$\gamma_{yz}^p = \gamma_{zx}^p = \gamma_{xy}^p = 0 \qquad \gamma_{yz}^e = \gamma_{zx}^e = \gamma_{xy}^e = 0$$

The second simple load path is pure shear along line OCD. Again the material yields at point C and strain hardens to point D. The permanent plastic strain and the resultant elastic strain at point D is

$$\varepsilon_x^p = \varepsilon_y^p = \varepsilon_z^p = \gamma_{yz}^p = \gamma_{zx}^p = 0 \qquad \varepsilon_x^e = \varepsilon_y^e = \varepsilon_z^e = \gamma_{yz}^e = \gamma_{zx}^e = 0$$

$$\gamma_{xy}^p = \gamma^p \qquad \gamma_{xy}^e = \gamma^e$$



The permanent plastic strains remaining in the tube are quite different for the two cases:

$$\varepsilon_x^p = \varepsilon^p \quad \varepsilon_y^p = \varepsilon_z^p = -\nu^p \varepsilon^p = -\frac{1}{2}\varepsilon^p \qquad \gamma_{yz}^p = \gamma_{zx}^p = \gamma_{xy}^p = 0$$

minus

$$\varepsilon_x^e = \varepsilon^e \quad \varepsilon_y^e = \varepsilon_z^e = -\nu^e \varepsilon^e \qquad \gamma_{yz}^e = \gamma_{zx}^e = \gamma_{xy}^e = 0$$

$$\varepsilon_x^p = \varepsilon_y^p = \varepsilon_z^p = \gamma_{yz}^p = \gamma_{zx}^p = 0 \qquad \gamma_{xy}^p = \gamma^p$$

minus

$$\varepsilon_x^e = \varepsilon_y^e = \varepsilon_z^e = \gamma_{yz}^e = \gamma_{zx}^e = 0 \qquad \gamma_{xy}^e = \gamma^e$$

However their effective stresses are equal (since they are on the same yield curve)

Thus they have yielded quite differently due to the different stress histories

The load paths considered are combinations of nonproportional stress components

Thus nonproportional loading leads to stress history dependence of the final strain state for a body.

The plastic deformation at point E can be calculated using deformation theory if the path OE is considered

In that case the plastic strains would be a mixture of normal and shear strains, which is very different from the unproportional paths both quantitatively and qualitatively.

Accordingly, we must recognize the dependence of our solution approach on the type of load path in a specific plasticity problem.

In reality the plastic strain is a function of both stresses and stress history.
So following the deformation in a step by step manner during loading process by the incremental theory gives the most accurate analysis of deformation.

$$\Delta\varepsilon = K(\sigma, \sigma_{HISTORY})\Delta\sigma \qquad d\varepsilon = K(\sigma, \sigma_{HISTORY})d\sigma$$

Where $K=1/E_{tan}$

Integrate the equation for a prescribed load path